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Baoding Liu

Uncertainty Theory

Fourth Edition

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Uncertainty Theory

Fourth Edition

Springer

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ISSN 2199-3807 ISSN 2199-3815 (electronic)
ISBN 978-3-662-44353-8 ISBN 978-3-662-44354-5 (eBook)
DOI 10.1007/978-3-662-44354-5

Library of Congress Control Number: 2014946387

Springer Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

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Preface

When no samples are available to estimate a probability distribution, we have to invite some domain experts to evaluate the belief degree that each event will happen. Perhaps some people think that the belief degree should be modeled by subjective probability or fuzzy set theory. However, it is usually inappropriate because both of them may lead to counterintuitive results in this case. In order to rationally deal with belief degrees, uncertainty theory was founded in 2007 and subsequently studied by many researchers. Nowadays, uncertainty theory has become a branch of axiomatic mathematics for modeling belief degrees.

Uncertain Measure

The most fundamental concept is uncertain measure that is a type of set function satisfying the axioms of uncertainty theory. It is used to indicate the belief degree that an uncertain event may happen. Chapter 1 will introduce normality, duality, subadditivity and product axioms. From those four axioms, this chapter will also present uncertain measure, product uncertain measure, and conditional uncertain measure.

Uncertain Variable

Uncertain variable is a measurable function from an uncertainty space to the set of real numbers. It is used to represent quantities with uncertainty. Chapter 2 is devoted to uncertain variable, uncertainty distribution, independence, operational law, expected value, variance, moments, entropy, distance, conditional uncertainty distribution, uncertain sequence, and uncertain vector.

Uncertain Programming

Uncertain programming is a type of mathematical programming involving uncertain variables. Chapter 3 will provide a type of uncertain programming model with applications to machine scheduling problem, vehicle routing problem, and project scheduling problem. In addition, uncertain multiobjective programming, uncertain goal programming and uncertain multilevel programming are also documented.

Uncertain Statistics

Uncertain statistics is a methodology for collecting and interpreting expert's experimental data by uncertainty theory. Chapter 4 will present a questionnaire survey for collecting expert's experimental data. In order to determine uncertainty distributions from those expert's experimental data, Chapter 4 will also introduce empirical uncertainty distribution, principle of least squares, method of moments, and Delphi method.

Uncertain Risk Analysis

The term risk has been used in different ways in literature. In this book the risk is defined as the accidental loss plus the uncertain measure of such loss, and a risk index is defined as the uncertain measure that some specified loss occurs. Chapter 5 will introduce uncertain risk analysis that is a tool to quantify risk via uncertainty theory. As applications of uncertain risk analysis, Chapter 5 will also discuss structural risk analysis and investment risk analysis.

Uncertain Reliability Analysis

Reliability index is defined as the uncertain measure that some system is working. Chapter 6 will introduce uncertain reliability analysis that is a tool to deal with system reliability via uncertainty theory.

Uncertain Propositional Logic

Uncertain propositional logic is a generalization of propositional logic in which every proposition is abstracted into a Boolean uncertain variable and the truth value is defined as the uncertain measure that the proposition is true. Chapter 7 will present uncertain propositional logic and uncertain predicate logic. In addition, uncertain entailment is a methodology for determining the truth value of an uncertain proposition via the maximum uncertainty principle when the truth values of other uncertain propositions are given. Chapter 8 will discuss an uncertain entailment model from which uncertain modus ponens, uncertain modus tollens and uncertain hypothetical syllogism are deduced.

Uncertain Set

Uncertain set is a set-valued function on an uncertainty space, and attempts to model "unsharp concepts". The main difference between uncertain set and uncertain variable is that the former takes values of set and the latter takes values of point. Uncertain set theory will be introduced in Chapter 9. In order to determine membership functions, Chapter 9 will also provide some methods of uncertain statistics.

Uncertain Logic

Some knowledge in human brain is actually an uncertain set. This fact encourages us to design an uncertain logic that is a methodology for calculating the truth values of uncertain propositions via uncertain set theory. Uncertain logic may provide a flexible means for extracting linguistic summary from a collection of raw data. Chapter 10 will be devoted to uncertain logic and linguistic summarizer.

Uncertain Inference

Uncertain inference is a process of deriving consequences from human knowledge via uncertain set theory. Chapter 11 will present a set of uncertain inference rules, uncertain system, and uncertain control with application to an inverted pendulum system.

Uncertain Process

An uncertain process is essentially a sequence of uncertain variables indexed by time. Thus an uncertain process is usually used to model uncertain phenomena that vary with time. Chapter 12 is devoted to basic concepts of uncertain process and uncertainty distribution. In addition, extreme value theorem, first hitting time and time integral of uncertain processes are also introduced. Chapter 13 deals with uncertain renewal process, renewal reward process, and alternating renewal process. Chapter 13 also provides block replacement policy, age replacement policy, and an uncertain insurance model.

Uncertain Calculus

Uncertain calculus is a branch of mathematics that deals with differentiation and integration of uncertain processes. Chapter 14 will introduce Liu process that is a stationary independent increment process whose increments are normal uncertain variables, and discuss Liu integral that is a type of uncertain integral with respect to Liu process. In addition, the fundamental theorem of uncertain calculus will be proved in this chapter from which the techniques of chain rule, change of variables, and integration by parts are also derived.

Uncertain Differential Equation

Uncertain differential equation is a type of differential equation involving uncertain processes. Chapter 15 will discuss the existence, uniqueness and stability of solutions of uncertain differential equations, and will introduce Yao-Chen formula that represents the solution of an uncertain differential equation by a family of solutions of ordinary differential equations. On the basis of this formula, some formulas to calculate extreme value, first hitting time, and time integral of solution are provided. Furthermore, some numerical methods for solving general uncertain differential equations are designed.

Uncertain Finance

As applications of uncertain differential equation, Chapter 16 will discuss uncertain stock model, uncertain interest rate model, and uncertain currency model.

Law of Truth Conservation

The law of excluded middle tells us that a proposition is either true or false, and the law of contradiction tells us that a proposition cannot be both true and false. In the state of indeterminacy, some people said, the law of excluded middle and the law of contradiction are no longer valid because the truth degree of a proposition is no longer 0 or 1. I cannot gainsay this viewpoint to a certain extent. But it does not mean that you might "go as you please". *The truth values of a proposition and its negation should sum to unity.* This is the law of truth conservation that is weaker than the law of excluded middle and the law of contradiction. Furthermore, the law of truth conservation agrees with the law of excluded middle and the law of contradiction when the uncertainty vanishes.

Maximum Uncertainty Principle

An event has no uncertainty if its uncertain measure is 1 because we may believe that the event happens. An event has no uncertainty too if its uncertain measure is 0 because we may believe that the event does not happen. An event is the most uncertain if its uncertain measure is 0.5 because the event and its complement may be regarded as "equally likely". In practice, if there is no information about the uncertain measure of an event, we should assign 0.5 to it. Sometimes, only partial information is available. In this case, the value of uncertain measure may be specified in some range. What value does the uncertain measure take? *For any event, if there are multiple reasonable values that an uncertain measure may take, then the value as close to 0.5 as possible is assigned to the event.* This is the maximum uncertainty principle.

Matlab Uncertainty Toolbox

Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) is a collection of functions built on Matlab for many methods of uncertainty theory, including uncertain programming, uncertain statistics, uncertain risk analysis, uncertain reliability analysis, uncertain logic, uncertain inference, uncertain differential equation, scheduling, logistics, data mining, control, and finance.

Lecture Slides

If you need lecture slides for uncertainty theory, please download them from the website at http://orsc.edu.cn/liu/resources.htm.

Uncertainty Theory Online

If you want to read more papers related to uncertainty theory and applications, please visit the website at http://orsc.edu.cn/online.

Purpose

The purpose is to equip the readers with a branch of axiomatic mathematics to deal with belief degrees. The textbook is suitable for researchers, engineers, and students in the field of mathematics, information science, operations research, industrial engineering, computer science, artificial intelligence, automation, economics, and management science.

Acknowledgment

This work was supported by National Natural Science Foundation of China Grant No.61273044.

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May 2014

To My Wife Jinlan

Chapter 0

Introduction

Real decisions are usually made in the state of indeterminacy. For modeling indeterminacy, there exist two mathematical systems, one is probability theory (Kolmogorov, 1933) and the other is uncertainty theory (Liu, 2007). Probability is interpreted as frequency, while uncertainty is interpreted as personal belief degree.

What is indeterminacy? What is frequency? What is belief degree? This chapter will answer these questions, and show in what situation we should use probability theory and in what situation we should use uncertainty theory. Finally, it is concluded that a rational man behaves as if he used uncertainty theory.

0.1 Indeterminacy

By *indeterminacy* we mean the phenomena whose outcomes cannot be exactly predicted in advance. For example, we cannot exactly predict which face will appear before we toss dice. Thus "tossing dice" is a type of indeterminate phenomenon. As another example, we cannot exactly predict tomorrow's stock price. That is, "stock price" is also a type of indeterminate phenomenon. Some other instances of indeterminacy include "roulette wheel", "product lifetime", "market demand", "bridge strength", "travel distance", etc.

Indeterminacy is absolute, while determinacy is relative. This is the reason why we say real decisions are usually made in the state of indeterminacy. How to model indeterminacy is thus an important research subject in not only mathematics but also science and engineering.

In order to describe an indeterminate quantity, personally I think there exist only two ways, one is *frequency* generated by samples (i.e., historical data), and the other is *belief degree* evaluated by domain experts. Could you imagine a third way?

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5_1

0.2 Frequency

Assume we have collected a set of samples for some indeterminate quantity (e.g. stock price). By *cumulative frequency* we mean a function representing the percentage of all samples that fall into the left side of the current point. It is clear that the cumulative frequency looks like a step function in Figure 1, and will always have bigger values as the current point moves from the left to right.

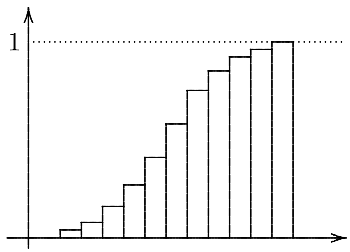

Figure 1: Cumulative frequency histogram

Frequency is a factual property of indeterminate quantity, and does not change with our state of knowledge and preference. In other words, the frequency in the long run exists and is relatively invariant, no matter if it is observed by us.

Probability theory is applicable when samples are available

The study of probability theory was started by Pascal and Fermat in the 17th century when they succeeded in deriving the exact probabilities for certain gambling problems. After that, probability theory was studied by many researchers. Particularly, a complete axiomatic foundation of probability theory was successfully given by Kolmogorov [88] in 1933. Since then, probability theory has been developed steadily and widely applied in science and engineering.

Keep in mind that a fundamental premise of applying probability theory is that the estimated probability distribution is close enough to the long-run cumulative frequency. Otherwise, the law of large numbers is no longer valid and probability theory is no longer applicable.

When the sample size is large enough, it is possible for us to believe the estimated probability distribution is close enough to the long-run cumulative frequency. In this case, there is no doubt that probability theory is the only legitimate approach to deal with our problems on the basis of the estimated probability distributions.

However, in many cases, no samples are available to estimate a probability distribution. What can we do in this situation? Perhaps we have no choice but to invite some domain experts to evaluate the belief degree that each event will happen.

0.3 Belief Degree

Belief degrees are familiar to all of us. The object of belief is an event (i.e., a proposition). For example, “the sun will rise tomorrow”, “it will be sunny next week”, and “John is a young man” are all instances of object of belief. A *belief degree* represents the strength with which we believe the event will happen. If we completely believe the event will happen, then the belief degree is 1 (complete belief). If we think it is completely impossible, then the belief degree is 0 (complete disbelief). If the event and its complementary event are equally likely, then the belief degree for the event is 0.5, and that for the complementary event is also 0.5. Generally, we will assign a number between 0 and 1 to the belief degree for each event. The higher the belief degree is, the more strongly we believe the event will happen.

Assume a box contains 100 balls, each of which is known to be either red or black, but we do not know how many of the balls are red and how many are black. In this case, it is impossible for us to determine the probability of drawing a red ball. However, the belief degree can be evaluated by us. For example, the belief degree for drawing a red ball is 0.5 because “drawing a red ball” and “drawing a black ball” are equally likely. Besides, the belief degree for drawing a black ball is also 0.5.

The belief degree depends heavily on the personal knowledge (even including preference) concerning the event. When the personal knowledge changes, the belief degree changes too.

Belief Degree Function

How do we describe an indeterminate quantity (e.g. bridge strength)? It is clear that a single belief degree is absolutely not enough. Do we need to know the belief degrees for all possible events? The answer is negative. In fact, what we need is a *belief degree function* that represents the degree with which we believe the indeterminate quantity falls into the left side of the current point.

For example, if we believe the indeterminate quantity completely falls into the left side of the current point, then the belief degree function takes value 1; if we think it completely falls into the right side, then the belief degree function takes value 0. Generally, a belief degree function takes values between 0 and 1, and has bigger values as the current point moves from the left to right. See Figure 2.

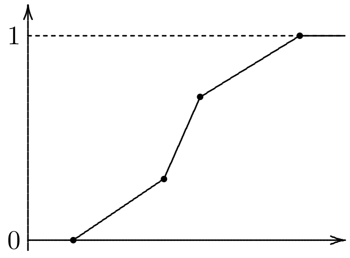

Figure 2: Belief degree function

How to obtain belief degrees

Consider a bridge and its strength. At first, we have to admit that no destructive experiment is allowed for the bridge. Thus we have no samples about the bridge strength. In this case, there do not exist any statistical methods to estimate its probability distribution. How do we deal with it? It seems that we have no choice but to invite some bridge engineers to evaluate the belief degrees about the bridge strength. In practice, it is almost impossible for the bridge engineers to give a perfect description of the belief degrees of all possible events. Instead, they can only provide some subjective judgments about the bridge strength. As a simple example, we assume a consultation process is as follows:

(Q) What do you think is the bridge strength?
(A) I think the bridge strength is between 80 and 120 tons.

What belief degrees can we derive from the answer of the bridge engineer? First, we may have an inference:

(i) I am 100% sure that the bridge strength is less than 120 tons.

This means the belief degree of "the bridge strength being less than 120 tons" is 1. Thus we have an expert's experimental data $(120, 1)$. Furthermore, we may have another inference:

(ii) I am 100% sure that the bridge strength is greater than 80 tons.

This statement gives a belief degree that the bridge strength falls into the right side of 80 tons. We need translate it to a statement about the belief degree that the bridge strength falls into the left side of 80 tons:

(ii′) I am 0% sure that the bridge strength is less than 80 tons.

Although the statement (ii′) sounds strange to us, it is indeed equivalent to the statement (ii). Thus we have another expert's experimental data $(80, 0)$.

Until now we have acquired two expert's experimental data $(80, 0)$ and $(120, 1)$ about the bridge strength. Could we infer the belief degree $\Phi(x)$

that the bridge strength falls into the left side of the point x? The answer is affirmative. For example, a reasonable value is

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 80 \\ (x-80)/40, & \text{if } 80 \le x \le 120 \\ 1, & \text{if } x > 120. \end{cases} \tag{1}$$

See Figure 3. From the function $\Phi(x)$, we may infer that the belief degree of "the bridge strength being less than 90 tons" is 0.25. In other words, it is reasonable to infer that "I am 25% sure that the bridge strength is less than 90 tons", or equivalently "I am 75% sure that the bridge strength is greater than 90 tons".

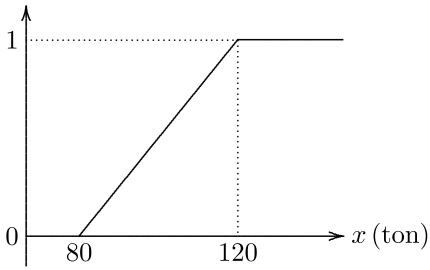

Figure 3: Belief degree function of "the bridge strength"

All belief degrees are wrong, but some are useful

Different people may produce different belief degrees. Perhaps some readers may ask which belief degree is correct. I would like to answer it in this way: *All belief degrees are wrong, but some are useful.* A belief degree becomes "correct" only when it is close enough to the frequency of the indeterminate quantity. However, usually we cannot make it to that.

Numerous surveys showed that *human beings usually estimate a much wider range of values than the object actually takes.* This conservatism of human beings makes the belief degrees deviate far from the frequency. Thus all belief degrees are wrong compared with its frequency. However, it cannot be denied that those belief degrees are indeed helpful for decision making.

Belief degrees cannot be treated as subjective probability

Can we deal with belief degrees by probability theory? Some people do think so and call it subjective probability. However, Liu [131] declared that it is inappropriate to model belief degrees by probability theory because it may lead to counterintuitive results.

Consider a counterexample presented by Liu [131]. Assume there is one truck and 50 bridges in an experiment. Also assume the weight of the truck is 90 tons and the 50 bridge strengths are iid uniform random variables on $[95, 110]$ in tons. For simplicity, suppose a bridge collapses whenever its real strength is less than the weight of the truck. Now let us have the truck cross over the 50 bridges one by one. It is easy to verify that

$$\Pr\{\text{“the truck can cross over the 50 bridges”}\} = 1. \tag{2}$$

That is to say, we are 100% sure that the truck can cross over the 50 bridges successfully.

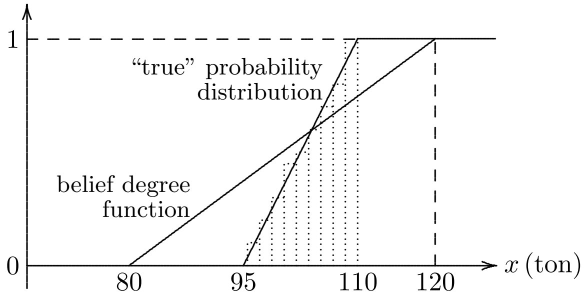

Figure 4: Belief degree function, “true” probability distribution and cumulative frequency histogram of “the bridge strength”

However, when there do not exist any observed samples for the bridge strength at the moment, we have to invite some bridge engineers to evaluate the belief degrees about it. As we stated before, human beings usually estimate a much wider range of values than the bridge strength actually takes because of the conservatism. Assume the belief degree function is

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 80 \\ (x-80)/40, & \text{if } 80 \le x \le 120 \\ 1, & \text{if } x > 120. \end{cases} \tag{3}$$

See Figure 4. Let us imagine what will happen if the belief degree function is treated as a probability distribution. At first, we have to regard the 50 bridge strengths as iid uniform random variables on $[80, 120]$ in tons. If we have the truck cross over the 50 bridges one by one, then we immediately have

$$\Pr\{\text{“the truck can cross over the 50 bridges”}\} = 0.75^{50} \approx 0. \tag{4}$$

Thus it is almost impossible that the truck crosses over the 50 bridges successfully. Unfortunately, the results (2) and (4) are at opposite poles. This example shows that, by inappropriately using probability theory, a sure event becomes an impossible one. The error seems intolerable for us. Hence the belief degrees cannot be treated as subjective probability.

A possible proposition cannot be judged impossible

During information processing, we should follow such a basic principle that a possible proposition cannot be judged impossible (Liu [131]). In other words, if a proposition is possibly true, then its truth value should not be zero. Likewise, if a proposition is possibly false, then its truth value should not be unity.

In the example of truck-cross-over-bridge, a completely true proposition is judged completely false. This means using probability theory violates the above-mentioned principle, and therefore probability theory is not appropriate to model belief degrees. In other words, belief degrees do not follow the laws of probability theory.

Uncertainty theory is able to model belief degrees

In order to rationally deal with belief degrees, uncertainty theory was founded by Liu [122] in 2007 and subsequently studied by many researchers. Nowadays, uncertainty theory has become a branch of axiomatic mathematics for modeling belief degrees.

Liu [131] declared that uncertainty theory is the only legitimate approach when only belief degrees are available. If we believe the estimated uncertainty distribution is close enough to the belief degrees hidden in the mind of the domain experts, then we may use uncertainty theory to deal with our own problems on the basis of the estimated uncertainty distributions.

Let us reconsider the example of truck-cross-over-bridge by uncertainty theory. If the belief degree function is regarded as a linear uncertainty distribution on $[80, 120]$ in tons, then we immediately have

$$\mathcal{M}\{\text{``the truck can cross over the 50 bridges''}\} = 0.75. \tag{5}$$

That is to say, we are 75% sure that the truck can cross over the 50 bridges successfully. Here the degree 75% does not achieve up to the true value 100%. But the error is caused by the difference between belief degree and frequency, and is not further magnified by uncertainty theory.

0.4 Summary

In order to model indeterminacy, many theories have been invented. What theories are considered acceptable? Personally, I think that *an acceptable theory should be not only theoretically self-consistent but also the best among others for solving at least one practical problem.* On the basis of this principle, I may conclude that there exist two mathematical systems, one is probability theory and the other is uncertainty theory. It is emphasized that probability theory is only applicable to modeling frequencies, and uncertainty theory is only applicable to modeling belief degrees. In other words, frequency is the empirical basis of probability theory, while belief degree is the empirical basis of uncertainty theory. Keep in mind that using uncertainty theory to model frequency may produce a crude result, while using probability theory to model belief degree may produce a big disaster.

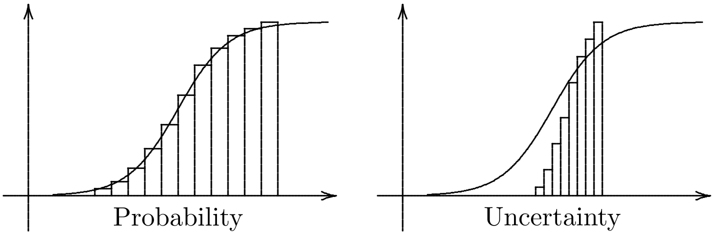

Figure 5: When the sample size is large enough, the estimated probability distribution (left curve) may be close enough to the cumulative frequency (left histogram). In this case, probability theory is the only legitimate approach. When the belief degrees are available (no samples), the estimated uncertainty distribution (right curve) usually deviates far from the cumulative frequency (right histogram but unknown). In this case, uncertainty theory is the only legitimate approach.

However, single-variable system is an exception. When there exists one and only one variable in a system, probability theory and uncertainty theory will produce the same result because product measure is not used. In this case, frequency may be modeled by uncertainty theory while belief degree may be modeled by probability theory. Both are indifferent.

Since belief degrees are usually wrong compared with frequency, the gap between belief degree and frequency always exists. Such an error is likely to be further magnified if the belief degree is regarded as subjective probability. Fortunately, uncertainty theory can successfully avoid turning small errors to large ones.

Savage [203] said a rational man behaves as if he used subjective probabilities. However, usually, we cannot make it to that. Personally, I think *a rational man behaves as if he used uncertainty theory.* In other words, a rational man is expected to hold belief degrees that follow the laws of uncertainty theory rather than probability theory.

Chapter 1

Uncertain Measure

Uncertainty theory was founded by Liu [122] in 2007 and subsequently studied by many researchers. Nowadays uncertainty theory has become a branch of axiomatic mathematics for modeling belief degrees. This chapter will present normality, duality, subadditivity and product axioms of uncertainty theory. From those four axioms, this chapter will also introduce an uncertain measure that is a fundamental concept in uncertainty theory. In addition, product uncertain measure and conditional uncertain measure will be explored at the end of this chapter.

1.1 Measurable Space

From the mathematical viewpoint, uncertainty theory is essentially an alternative theory of measure. Thus uncertainty theory should begin with a measurable space. In order to learn uncertainty theory, let us introduce algebra, σ-algebra, measurable set, Borel algebra, Borel set, and measurable function. The main results in this section are well-known. For this reason the credit references are not provided. You may skip this section if you are familiar with them.

Definition 1.1 *Let Γ be a nonempty set (sometimes called universal set). A collection $\mathcal{L}$ consisting of subsets of Γ is called an algebra over Γ if the following three conditions hold: (a) $\Gamma \in \mathcal{L}$; (b) if $\Lambda \in \mathcal{L}$, then $\Lambda^c \in \mathcal{L}$; and (c) if $\Lambda_1, \Lambda_2, \cdots, \Lambda_n \in \mathcal{L}$, then*

$$\bigcup_{i=1}^{n} \Lambda_i \in \mathcal{L}. \tag{1.1}$$

The collection $\mathcal{L}$ is called a σ-algebra over Γ if the condition (c) is replaced

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5_2

with closure under countable union, i.e., when $\Lambda_1, \Lambda_2, \cdots \in \mathcal{L}$, we have

$$\bigcup_{i=1}^{\infty} \Lambda_i \in \mathcal{L}. \tag{1.2}$$

Example 1.1: The collection $\{\emptyset, \Gamma\}$ is the smallest σ-algebra over Γ, and the power set (i.e., all subsets of Γ) is the largest σ-algebra.

Example 1.2: Let Λ be a proper nonempty subset of Γ. Then $\{\emptyset, \Lambda, \Lambda^c, \Gamma\}$ is a σ-algebra over Γ.

Example 1.3: Let $\mathcal{L}$ be the collection of all finite disjoint unions of all intervals of the form

$$(-\infty, a], \quad (a, b], \quad (b, \infty), \quad \emptyset. \tag{1.3}$$

Then $\mathcal{L}$ is an algebra over $\Re$ (the set of real numbers), but not a σ-algebra because $\Lambda_i = (0, (i-1)/i] \in \mathcal{L}$ for all i but

$$\bigcup_{i=1}^{\infty} \Lambda_i = (0, 1) \not\in \mathcal{L}. \tag{1.4}$$

Example 1.4: A σ-algebra $\mathcal{L}$ is closed under countable union, countable intersection, difference, and limit. That is, if $\Lambda_1, \Lambda_2, \cdots \in \mathcal{L}$, then

$$\bigcup_{i=1}^{\infty} \Lambda_i \in \mathcal{L}; \quad \bigcap_{i=1}^{\infty} \Lambda_i \in \mathcal{L}; \quad \Lambda_1 \setminus \Lambda_2 \in \mathcal{L}; \quad \lim_{i \to \infty} \Lambda_i \in \mathcal{L}. \tag{1.5}$$

Definition 1.2 *Let Γ be a nonempty set, and let $\mathcal{L}$ be a σ-algebra over Γ. Then $(\Gamma, \mathcal{L})$ is called a measurable space, and any element in $\mathcal{L}$ is called a measurable set.*

Example 1.5: Let $\Re$ be the set of real numbers. Then $\mathcal{L} = \{\emptyset, \Re\}$ is a σ-algebra over $\Re$. Thus $(\Re, \mathcal{L})$ is a measurable space. Note that there exist only two measurable sets in this space, one is $\emptyset$ and another is $\Re$. Keep in mind that the intervals like $[0, 1]$ and $(0, +\infty)$ are not measurable!

Example 1.6: Let $\Gamma = \{a, b, c\}$. Then $\mathcal{L} = \{\emptyset, \{a\}, \{b, c\}, \Gamma\}$ is a σ-algebra over Γ. Thus $(\Gamma, \mathcal{L})$ is a measurable space. Furthermore, $\{a\}$ and $\{b, c\}$ are measurable sets in this space, but $\{b\}, \{c\}, \{a, b\}, \{a, c\}$ are not.

Definition 1.3 *The smallest σ-algebra $\mathcal{B}$ containing all open intervals is called the Borel algebra over the set of real numbers, and any element in $\mathcal{B}$ is called a Borel set.*

Example 1.7: It has been proved that intervals, open sets, closed sets, rational numbers, and irrational numbers are all Borel sets.

Example 1.8: There exists a non-Borel set over $\Re$. Let $[a]$ represent the set of all rational numbers plus a. Note that if $a_1 - a_2$ is not a rational number, then $[a_1]$ and $[a_2]$ are disjoint sets. Thus $\Re$ is divided into an infinite number of those disjoint sets. Let A be a new set containing precisely one element from them. Then A is not a Borel set.

Definition 1.4 *A function f from a measurable space $(\Gamma, \mathcal{L})$ to the set of real numbers is said to be measurable if*

$$f^{-1}(B) = \{\gamma \in \Gamma \mid f(\gamma) \in B\} \in \mathcal{L} \tag{1.6}$$

for any Borel set B of real numbers.

Continuous function and monotone function are instances of measurable function. Let $f_1, f_2, \cdots$ be a sequence of measurable functions. Then the following functions are also measurable:

$$\sup_{1 \le i < \infty} f_i(\gamma); \quad \inf_{1 \le i < \infty} f_i(\gamma); \quad \limsup_{i \to \infty} f_i(\gamma); \quad \liminf_{i \to \infty} f_i(\gamma). \tag{1.7}$$

Especially, if $\lim_{i \to \infty} f_i(\gamma)$ exists for each γ, then the limit is also a measurable function.

1.2 Event

Let $(\Gamma, \mathcal{L})$ be a measurable space. Recall that each element Λ in $\mathcal{L}$ is called a measurable set. The first action we take is to rename measurable set as *event* in uncertainty theory.

How do we understand those terminologies? Let us illustrate them by an indeterminate quantity (e.g. bridge strength). At first, the universal set Γ consists of all possible outcomes of the indeterminate quantity. If we believe that the possible bridge strengths range from 80 to 120 in tons, then the universal set is

$$\Gamma = [80, 120]. \tag{1.8}$$

Note that you may replace the universal set with an enlarged interval, and it would have no impact.

The σ-algebra $\mathcal{L}$ should contain all events we are concerned about. Note that event and proposition are synonymous although the former is a set and the latter is a statement. Assume the first event we are concerned about corresponds to the proposition "the bridge strength is less than or equal to 100 tons". Then it may be represented by

$$\Lambda_1 = [80, 100]. \tag{1.9}$$

Also assume the second event we are concerned about corresponds to the proposition “the bridge strength is more than 100 tons”. Then it may be represented by

$$\Lambda_2 = (100, 120]. \tag{1.10}$$

If we are only concerned about the above two events, then we may construct a σ-algebra $\mathcal{L}$ containing the two events Λ_1 and Λ_2, for example,

$$\mathcal{L} = \{\emptyset,\, \Lambda_1,\, \Lambda_2,\, \Gamma\}. \tag{1.11}$$

In this case, we totally have four events: $\emptyset$, Λ_1, Λ_2 and Γ. However, please note that the subsets like $[80, 90]$ and $[110, 120]$ are not events because they do not belong to $\mathcal{L}$.

Keep in mind that different σ-algebras are used for different purposes. The minimum requirement of a σ-algebra is that it contains all events we are concerned about. It is suggested to take the minimum σ-algebra that contains those events.

1.3 Uncertain Measure

Let us define an uncertain measure $\mathcal{M}$ on the σ-algebra $\mathcal{L}$. That is, a number $\mathcal{M}\{\Lambda\}$ will be assigned to each event Λ to indicate the belief degree with which we believe Λ will happen. There is no doubt that the assignment is not arbitrary, and the uncertain measure $\mathcal{M}$ must have certain mathematical properties. In order to rationally deal with belief degrees, Liu [122] suggested the following three axioms:

Axiom 1. *(Normality Axiom)* $\mathcal{M}\{\Gamma\} = 1$ *for the universal set* Γ.

Axiom 2. *(Duality Axiom)* $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any event* Λ.

Axiom 3. *(Subadditivity Axiom) For every countable sequence of events* Λ_1, $\Lambda_2, \cdots$, *we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}. \tag{1.12}$$

Remark 1.1: Uncertain measure is interpreted as the personal belief degree (not frequency) of an uncertain event that may happen. It depends on the personal knowledge concerning the event. The uncertain measure will change if the state of knowledge changes.

Remark 1.2: Duality axiom is in fact an application of the law of truth conservation in uncertainty theory. The property ensures that the uncertainty theory is consistent with the law of excluded middle and the law of contradiction. In addition, the human thinking is always dominated by the duality. For example, if someone says a proposition is true with belief degree

0.6, then all of us will think that the proposition is false with belief degree 0.4.

Remark 1.3: Given two events with known belief degrees, it is frequently asked that how the belief degree for their union is generated from the individuals. Personally, I do not think there exists any rule to make it. A lot of surveys showed that, generally speaking, the belief degree of a union of events is neither the sum of belief degrees of the individual events (e.g. probability measure) nor the maximum (e.g. possibility measure). Perhaps there is no explicit relation between the union and individuals except for the subadditivity axiom.

Remark 1.4: Pathology occurs if subadditivity axiom is not assumed. For example, suppose that a universal set contains 3 elements. We define a set function that takes value 0 for each singleton, and 1 for each event with at least 2 elements. Then such a set function satisfies all axioms but subadditivity. Do you think it is strange if such a set function serves as a measure?

Remark 1.5: Although probability measure satisfies the above three axioms, probability theory is not a special case of uncertainty theory because the product probability measure does not satisfy the fourth axiom, namely the product axiom on Page 17.

Definition 1.5 *(Liu [122]) The set function $\mathcal{M}$ is called an uncertain measure if it satisfies the normality, duality, and subadditivity axioms.*

Exercise 1.1: Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$. It is clear that there exist 8 events in the σ-algebra

$$\mathcal{L} = \{\emptyset, \{\gamma_1\}, \{\gamma_2\}, \{\gamma_3\}, \{\gamma_1, \gamma_2\}, \{\gamma_1, \gamma_3\}, \{\gamma_2, \gamma_3\}, \Gamma\}. \tag{1.13}$$

Assume c_1, c_2, c_3 are nonnegative numbers satisfying the consistency condition

$$c_i + c_j \le 1 \le c_1 + c_2 + c_3, \quad \forall i \ne j. \tag{1.14}$$

Define

$$\mathcal{M}\{\gamma_1\} = c_1, \quad \mathcal{M}\{\gamma_2\} = c_2, \quad \mathcal{M}\{\gamma_3\} = c_3,$$

$$\mathcal{M}\{\gamma_1, \gamma_2\} = 1 - c_3, \quad \mathcal{M}\{\gamma_1, \gamma_3\} = 1 - c_2, \quad \mathcal{M}\{\gamma_2, \gamma_3\} = 1 - c_1,$$

$$\mathcal{M}\{\emptyset\} = 0, \quad \mathcal{M}\{\Gamma\} = 1.$$

Show that $\mathcal{M}$ is an uncertain measure.

Exercise 1.2: Suppose that $\lambda(x)$ is a nonnegative function on $\Re$ (the set of real numbers) such that

$$\sup_{x \in \Re} \lambda(x) = 0.5. \tag{1.15}$$

Define a set function

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{x\in\Lambda} \lambda(x), & \text{if } \sup\limits_{x\in\Lambda} \lambda(x) < 0.5 \\ 1 - \sup\limits_{x\in\Lambda^c} \lambda(x), & \text{if } \sup\limits_{x\in\Lambda} \lambda(x) = 0.5 \end{cases} \tag{1.16}$$

for each Borel set Λ. Show that $\mathcal{M}$ is an uncertain measure on $\Re$.

Exercise 1.3: Suppose $\rho(x)$ is a nonnegative and integrable function on $\Re$ (the set of real numbers) such that

$$\int_{\Re} \rho(x)\mathrm{d}x \geq 1. \tag{1.17}$$

Define a set function

$$\mathcal{M}\{\Lambda\} = \begin{cases} \displaystyle\int_{\Lambda} \rho(x)\mathrm{d}x, & \text{if } \displaystyle\int_{\Lambda} \rho(x)\mathrm{d}x < 0.5 \\ 1 - \displaystyle\int_{\Lambda^c} \rho(x)\mathrm{d}x, & \text{if } \displaystyle\int_{\Lambda^c} \rho(x)\mathrm{d}x < 0.5 \\ 0.5, & \text{otherwise} \end{cases} \tag{1.18}$$

for each Borel set Λ. Show that $\mathcal{M}$ is an uncertain measure on $\Re$.

Theorem 1.1 *(Monotonicity Theorem) Uncertain measure $\mathcal{M}$ is a monotone increasing set function. That is, for any events $\Lambda_1 \subset \Lambda_2$, we have*

$$\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}. \tag{1.19}$$

Proof: The normality axiom says $\mathcal{M}\{\Gamma\} = 1$, and the duality axiom says $\mathcal{M}\{\Lambda_1^c\} = 1 - \mathcal{M}\{\Lambda_1\}$. Since $\Lambda_1 \subset \Lambda_2$, we have $\Gamma = \Lambda_1^c \cup \Lambda_2$. By using the subadditivity axiom, we obtain

$$1 = \mathcal{M}\{\Gamma\} \leq \mathcal{M}\{\Lambda_1^c\} + \mathcal{M}\{\Lambda_2\} = 1 - \mathcal{M}\{\Lambda_1\} + \mathcal{M}\{\Lambda_2\}.$$

Thus $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$.

Theorem 1.2 *Suppose that $\mathcal{M}$ is an uncertain measure. Then the empty set $\emptyset$ has an uncertain measure zero, i.e.,*

$$\mathcal{M}\{\emptyset\} = 0. \tag{1.20}$$

Proof: Since $\emptyset = \Gamma^c$ and $\mathcal{M}\{\Gamma\} = 1$, it follows from the duality axiom that

$$\mathcal{M}\{\emptyset\} = 1 - \mathcal{M}\{\Gamma\} = 1 - 1 = 0.$$

Theorem 1.3 *Suppose that $\mathcal{M}$ is an uncertain measure. Then for any event Λ, we have*

$$0 \leq \mathcal{M}\{\Lambda\} \leq 1. \tag{1.21}$$

Proof: It follows from the monotonicity theorem that $0 \le \mathcal{M}\{\Lambda\} \le 1$ because $\emptyset \subset \Lambda \subset \Gamma$ and $\mathcal{M}\{\emptyset\} = 0$, $\mathcal{M}\{\Gamma\} = 1$.

Theorem 1.4 *Let $\Lambda_1, \Lambda_2, \cdots$ be a sequence of events with $\mathcal{M}\{\Lambda_i\} \to 0$ as $i \to \infty$. Then for any event Λ, we have*

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda \cup \Lambda_i\} = \lim_{i\to\infty} \mathcal{M}\{\Lambda \backslash \Lambda_i\} = \mathcal{M}\{\Lambda\}. \tag{1.22}$$

Especially, an uncertain measure remains unchanged if the event is enlarged or reduced by an event with uncertain measure zero.

Proof: It follows from the monotonicity theorem and subadditivity axiom that

$$\mathcal{M}\{\Lambda\} \le \mathcal{M}\{\Lambda \cup \Lambda_i\} \le \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda_i\}$$

for each i. Thus we get $\mathcal{M}\{\Lambda \cup \Lambda_i\} \to \mathcal{M}\{\Lambda\}$ by using $\mathcal{M}\{\Lambda_i\} \to 0$. Since $(\Lambda \backslash \Lambda_i) \subset \Lambda \subset ((\Lambda \backslash \Lambda_i) \cup \Lambda_i)$, we have

$$\mathcal{M}\{\Lambda \backslash \Lambda_i\} \le \mathcal{M}\{\Lambda\} \le \mathcal{M}\{\Lambda \backslash \Lambda_i\} + \mathcal{M}\{\Lambda_i\}.$$

Hence $\mathcal{M}\{\Lambda \backslash \Lambda_i\} \to \mathcal{M}\{\Lambda\}$ by using $\mathcal{M}\{\Lambda_i\} \to 0$.

Theorem 1.5 *(Asymptotic Theorem) For any events $\Lambda_1, \Lambda_2, \cdots$, we have*

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} > 0, \quad \text{if } \Lambda_i \uparrow \Gamma, \tag{1.23}$$

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} < 1, \quad \text{if } \Lambda_i \downarrow \emptyset. \tag{1.24}$$

Proof: Assume $\Lambda_i \uparrow \Gamma$. Since $\Gamma = \cup_i \Lambda_i$, it follows from the subadditivity axiom that

$$1 = \mathcal{M}\{\Gamma\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Since $\mathcal{M}\{\Lambda_i\}$ is increasing with respect to i, we have $\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} > 0$. If $\Lambda_i \downarrow \emptyset$, then $\Lambda_i^c \uparrow \Gamma$. It follows from the first inequality and the duality axiom that

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} = 1 - \lim_{i\to\infty} \mathcal{M}\{\Lambda_i^c\} < 1.$$

The theorem is proved.

Example 1.9: Assume Γ is the set of real numbers. Let α be a number with $0 < \alpha \le 0.5$. Define a set function as follows,

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ \alpha, & \text{if } \Lambda \text{ is upper bounded} \\ 0.5, & \text{if both } \Lambda \text{ and } \Lambda^c \text{ are upper unbounded} \\ 1-\alpha, & \text{if } \Lambda^c \text{ is upper bounded} \\ 1, & \text{if } \Lambda = \Gamma. \end{cases} \tag{1.25}$$

It is easy to verify that $\mathcal{M}$ is an uncertain measure. Write $\Lambda_i = (-\infty, i]$ for $i = 1, 2, \cdots$ Then $\Lambda_i \uparrow \Gamma$ and $\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} = \alpha$. Furthermore, we have $\Lambda_i^c \downarrow \emptyset$ and $\lim_{i\to\infty} \mathcal{M}\{\Lambda_i^c\} = 1 - \alpha$.

1.4 Uncertainty Space

Definition 1.6 *(Liu [122]) Let* Γ *be a nonempty set, let* $\mathcal{L}$ *be a* σ*-algebra over* Γ*, and let* $\mathcal{M}$ *be an uncertain measure. Then the triplet* $(\Gamma, \mathcal{L}, \mathcal{M})$ *is called an uncertainty space.*

For practical purposes, the study of uncertainty spaces is sometimes restricted to complete uncertainty spaces.

Definition 1.7 *An uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *is called complete if for any* $\Lambda_1, \Lambda_2 \in \mathcal{L}$ *with* $\mathcal{M}\{\Lambda_1\} = \mathcal{M}\{\Lambda_2\}$ *and any subset* A *with* $\Lambda_1 \subset A \subset \Lambda_2$*, one has* $A \in \mathcal{L}$*. In this case, we also have*

$$\mathcal{M}\{A\} = \mathcal{M}\{\Lambda_1\} = \mathcal{M}\{\Lambda_2\}. \tag{1.26}$$

Exercise 1.4: Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be a complete uncertainty space, and let Λ be an event with $\mathcal{M}\{\Lambda\} = 0$. Show that A is an event and $\mathcal{M}\{A\} = 0$ whenever $A \subset \Lambda$.

Exercise 1.5: Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be a complete uncertainty space, and let Λ be an event with $\mathcal{M}\{\Lambda\} = 1$. Show that A is an event and $\mathcal{M}\{A\} = 1$ whenever $A \supset \Lambda$.

Definition 1.8 *(Gao [48]) An uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *is called continuous if for any events* $\Lambda_1, \Lambda_2, \cdots$*, we have*

$$\mathcal{M}\left\{\lim_{i\to\infty} \Lambda_i\right\} = \lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} \tag{1.27}$$

provided that $\lim_{i\to\infty} \Lambda_i$ exists.

Exercise 1.6: Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be a continuous uncertainty space. For any events $\Lambda_1, \Lambda_2, \cdots$, show that

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} = 1, \quad \text{if } \Lambda_i \uparrow \Gamma, \tag{1.28}$$

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} = 0, \quad \text{if } \Lambda_i \downarrow \emptyset. \tag{1.29}$$

1.5 Product Uncertain Measure

Product uncertain measure was defined by Liu [125] in 2009, thus producing the fourth axiom of uncertainty theory. Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$ Write

$$\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \tag{1.30}$$

that is the set of all ordered tuples of the form $(\gamma_1, \gamma_2, \cdots)$, where $\gamma_k \in \Gamma_k$ for $k = 1, 2, \cdots$ A measurable rectangle in Γ is a set

$$\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots \tag{1.31}$$

where $\Lambda_k \in \mathcal{L}_k$ for $k = 1, 2, \cdots$ The smallest σ-algebra containing all measurable rectangles of Γ is called the product σ-algebra, denoted by

$$\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \tag{1.32}$$

Then the product uncertain measure $\mathcal{M}$ on the product σ-algebra $\mathcal{L}$ is defined by the following product axiom (Liu [125]).

Axiom 4. *(Product Axiom) Let* $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ *be uncertainty spaces for* $k = 1, 2, \cdots$ *The product uncertain measure* $\mathcal{M}$ *is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\} \tag{1.33}$$

where Λ_k *are arbitrarily chosen events from* $\mathcal{L}_k$ *for* $k = 1, 2, \cdots$*, respectively.*

Remark 1.6: Note that (1.33) defines a product uncertain measure only for rectangles. How do we extend the uncertain measure $\mathcal{M}$ from the class of rectangles to the product σ-algebra $\mathcal{L}$? For each event $\Lambda \in \mathcal{L}$, we have

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min\limits_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\}, \\ \qquad \text{if } \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min\limits_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 1 - \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min\limits_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\}, \\ \qquad \text{if } \sup\limits_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min\limits_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 0.5, \qquad \text{otherwise.} \end{cases} \tag{1.34}$$

Remark 1.7: Note that the sum of the uncertain measures of the maximum rectangles in Λ and Λ^c is always less than or equal to 1, i.e.,

$$\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} + \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} \le 1.$$

This means that at most one of

$$\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} \quad \text{and} \quad \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\}$$

is greater than 0.5. Thus the expression (1.34) is reasonable.

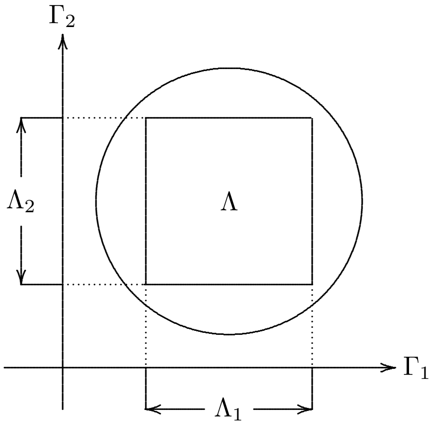

Figure 1.1: Extension from Rectangles to Product σ-Algebra. The uncertain measure of Λ (the disk) is essentially the acreage of its inscribed rectangle $\Lambda_1\times\Lambda_2$ if it is greater than 0.5. Otherwise, we have to examine its complement Λ^c. If the inscribed rectangle of Λ^c is greater than 0.5, then $\mathcal{M}\{\Lambda^c\}$ is just its inscribed rectangle and $\mathcal{M}\{\Lambda\} = 1 - \mathcal{M}\{\Lambda^c\}$. If there does not exist an inscribed rectangle of Λ or Λ^c greater than 0.5, then we set $\mathcal{M}\{\Lambda\} = 0.5$. Reprinted from Liu [129].

Remark 1.8: If the sum of the uncertain measures of the maximum rectangles in Λ and Λ^c is just 1, i.e.,

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}+\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}=1,$$

then the product uncertain measure (1.34) is simplified as

$$\mathcal{M}\{\Lambda\}=\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}. \tag{1.35}$$

Theorem 1.6 *(Peng and Iwamura [185]) The product uncertain measure defined by (1.34) is an uncertain measure.*

Proof: In order to prove that the product uncertain measure (1.34) is indeed an uncertain measure, we should verify that the product uncertain measure satisfies the normality, duality and subadditivity axioms.

Step 1: The product uncertain measure is clearly normal, i.e., $\mathcal{M}\{\Gamma\} = 1$.

Step 2: We prove the duality, i.e., $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$. The argument breaks down into three cases. Case 1: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}>0.5.$$

Then we immediately have

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}<0.5.$$

It follows from (1.34) that

$$\mathcal{M}\{\Lambda\}=\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\},$$

$$\mathcal{M}\{\Lambda^c\}=1-\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset(\Lambda^c)^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}=1-\mathcal{M}\{\Lambda\}.$$

The duality is proved. Case 2: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}>0.5.$$

This case may be proved by a similar process. Case 3: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\le 0.5$$

and

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\le 0.5.$$

It follows from (1.34) that $\mathcal{M}\{\Lambda\}=\mathcal{M}\{\Lambda^c\}=0.5$ which proves the duality.

Step 3: Let us prove that $\mathcal{M}$ is an increasing set function. Suppose Λ and Δ are two events in $\mathcal{L}$ with $\Lambda\subset\Delta$. The argument breaks down into three cases. Case 1: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}>0.5.$$

Then

$$\sup_{\Delta_1\times\Delta_2\times\cdots\subset\Delta}\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}\ge\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}>0.5.$$

It follows from (1.34) that $\mathcal{M}\{\Lambda\}\le\mathcal{M}\{\Delta\}$. Case 2: Assume

$$\sup_{\Delta_1\times\Delta_2\times\cdots\subset\Delta^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}>0.5.$$

Then

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\ge\sup_{\Delta_1\times\Delta_2\times\cdots\subset\Delta^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}>0.5.$$

Thus

$$\begin{aligned}\mathcal{M}\{\Lambda\}&=1-\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\\&\le 1-\sup_{\Delta_1\times\Delta_2\times\cdots\subset\Delta^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}=\mathcal{M}\{\Delta\}.\end{aligned}$$

Case 3: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\ \min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\le 0.5$$

and

$$\sup_{\Delta_1\times\Delta_2\times\cdots\subset\Delta^c}\ \min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}\le 0.5.$$

Then

$$\mathcal{M}\{\Lambda\}\le 0.5\le 1-\mathcal{M}\{\Delta^c\}=\mathcal{M}\{\Delta\}.$$

STEP 4: Finally, we prove the subadditivity of $\mathcal{M}$. For simplicity, we only prove the case of two events Λ and Δ. The argument breaks down into three cases. Case 1: Assume $\mathcal{M}\{\Lambda\}<0.5$ and $\mathcal{M}\{\Delta\}<0.5$. For any given $\varepsilon>0$, there are two rectangles

$$\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c,\quad \Delta_1\times\Delta_2\times\cdots\subset\Delta^c$$

such that

$$1-\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\le\mathcal{M}\{\Lambda\}+\varepsilon/2,$$
$$1-\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}\le\mathcal{M}\{\Delta\}+\varepsilon/2.$$

Note that

$$(\Lambda_1\cap\Delta_1)\times(\Lambda_2\cap\Delta_2)\times\cdots\subset(\Lambda\cup\Delta)^c.$$

It follows from the duality and subadditivity axioms that

$$\begin{aligned}\mathcal{M}_k\{\Lambda_k\cap\Delta_k\}&=1-\mathcal{M}_k\{(\Lambda_k\cap\Delta_k)^c\}=1-\mathcal{M}_k\{\Lambda_k^c\cup\Delta_k^c\}\\&\ge 1-(\mathcal{M}_k\{\Lambda_k^c\}+\mathcal{M}_k\{\Delta_k^c\})\\&=1-(1-\mathcal{M}_k\{\Lambda_k\})-(1-\mathcal{M}_k\{\Delta_k\})\\&=\mathcal{M}_k\{\Lambda_k\}+\mathcal{M}_k\{\Delta_k\}-1\end{aligned}$$

for any k. Thus

$$\begin{aligned}\mathcal{M}\{\Lambda\cup\Delta\}&\le 1-\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\cap\Delta_k\}\\&\le 1-\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}+1-\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}\\&\le\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Delta\}+\varepsilon.\end{aligned}$$

Letting $\varepsilon\to 0$, we obtain

$$\mathcal{M}\{\Lambda\cup\Delta\}\le\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Delta\}.$$

Case 2: Assume $\mathcal{M}\{\Lambda\}\ge 0.5$ and $\mathcal{M}\{\Delta\}<0.5$. When $\mathcal{M}\{\Lambda\cup\Delta\}=0.5$, the subadditivity is obvious. Now we consider the case $\mathcal{M}\{\Lambda\cup\Delta\}>0.5$, i.e., $\mathcal{M}\{\Lambda^c\cap\Delta^c\}<0.5$. By using $\Lambda^c\cup\Delta=(\Lambda^c\cap\Delta^c)\cup\Delta$ and Case 1, we get

$$\mathcal{M}\{\Lambda^c\cup\Delta\}\le\mathcal{M}\{\Lambda^c\cap\Delta^c\}+\mathcal{M}\{\Delta\}.$$

Thus

$$\mathcal{M}\{\Lambda\cup\Delta\} = 1-\mathcal{M}\{\Lambda^c\cap\Delta^c\} \le 1-\mathcal{M}\{\Lambda^c\cup\Delta\}+\mathcal{M}\{\Delta\}$$
$$\le 1-\mathcal{M}\{\Lambda^c\}+\mathcal{M}\{\Delta\} = \mathcal{M}\{\Lambda\}+\mathcal{M}\{\Delta\}.$$

Case 3: If both $\mathcal{M}\{\Lambda\}\ge 0.5$ and $\mathcal{M}\{\Delta\}\ge 0.5$, then the subadditivity is obvious because $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Delta\}\ge 1$. The theorem is proved.

Definition 1.9 *Assume* $(\Gamma_k,\mathcal{L}_k,\mathcal{M}_k)$ *are uncertainty spaces for* $k=1,2,\cdots$ *Let* $\Gamma=\Gamma_1\times\Gamma_2\times\cdots$, $\mathcal{L}=\mathcal{L}_1\times\mathcal{L}_2\times\cdots$ *and* $\mathcal{M}=\mathcal{M}_1\wedge\mathcal{M}_2\wedge\cdots$ *Then the triplet* $(\Gamma,\mathcal{L},\mathcal{M})$ *is called a product uncertainty space.*

1.6 Independence

Definition 1.10 *(Liu [129]) The events* $\Lambda_1,\Lambda_2,\cdots,\Lambda_n$ *are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}\Lambda_i^*\right\}=\bigwedge_{i=1}^{n}\mathcal{M}\{\Lambda_i^*\} \tag{1.36}$$

where Λ_i^* *are arbitrarily chosen from* $\{\Lambda_i,\Lambda_i^c,\Gamma\}$, $i=1,2,\cdots,n$, *respectively, and* Γ *is the sure event.*

Remark 1.9: Especially, two events Λ_1 and Λ_2 are independent if and only if

$$\mathcal{M}\left\{\Lambda_1^*\cap\Lambda_2^*\right\}=\mathcal{M}\{\Lambda_1^*\}\wedge\mathcal{M}\{\Lambda_2^*\} \tag{1.37}$$

where Λ_i^* are arbitrarily chosen from $\{\Lambda_i,\Lambda_i^c\}$, $i=1,2$, respectively. That is, the following four equations hold:

$$\mathcal{M}\{\Lambda_1\cap\Lambda_2\}=\mathcal{M}\{\Lambda_1\}\wedge\mathcal{M}\{\Lambda_2\},$$
$$\mathcal{M}\{\Lambda_1^c\cap\Lambda_2\}=\mathcal{M}\{\Lambda_1^c\}\wedge\mathcal{M}\{\Lambda_2\},$$
$$\mathcal{M}\{\Lambda_1\cap\Lambda_2^c\}=\mathcal{M}\{\Lambda_1\}\wedge\mathcal{M}\{\Lambda_2^c\},$$
$$\mathcal{M}\{\Lambda_1^c\cap\Lambda_2^c\}=\mathcal{M}\{\Lambda_1^c\}\wedge\mathcal{M}\{\Lambda_2^c\}.$$

Example 1.10: The impossible event $\emptyset$ is independent of any event Λ because $\emptyset^c=\Gamma$ and

$$\mathcal{M}\{\emptyset\cap\Lambda\}=\mathcal{M}\{\emptyset\}=\mathcal{M}\{\emptyset\}\wedge\mathcal{M}\{\Lambda\},$$
$$\mathcal{M}\{\emptyset^c\cap\Lambda\}=\mathcal{M}\{\Lambda\}=\mathcal{M}\{\emptyset^c\}\wedge\mathcal{M}\{\Lambda\},$$
$$\mathcal{M}\{\emptyset\cap\Lambda^c\}=\mathcal{M}\{\emptyset\}=\mathcal{M}\{\emptyset\}\wedge\mathcal{M}\{\Lambda^c\},$$
$$\mathcal{M}\{\emptyset^c\cap\Lambda^c\}=\mathcal{M}\{\Lambda^c\}=\mathcal{M}\{\emptyset^c\}\wedge\mathcal{M}\{\Lambda^c\}.$$

Example 1.11: The sure event Γ is independent of any event Λ because $\Gamma^c = \emptyset$ and

$$\begin{aligned}\mathcal{M}\{\Gamma \cap \Lambda\} = \mathcal{M}\{\Lambda\} = \mathcal{M}\{\Gamma\} \wedge \mathcal{M}\{\Lambda\},\\ \mathcal{M}\{\Gamma^c \cap \Lambda\} = \mathcal{M}\{\Gamma^c\} = \mathcal{M}\{\Gamma^c\} \wedge \mathcal{M}\{\Lambda\},\\ \mathcal{M}\{\Gamma \cap \Lambda^c\} = \mathcal{M}\{\Lambda^c\} = \mathcal{M}\{\Gamma\} \wedge \mathcal{M}\{\Lambda^c\},\\ \mathcal{M}\{\Gamma^c \cap \Lambda^c\} = \mathcal{M}\{\Gamma^c\} = \mathcal{M}\{\Gamma^c\} \wedge \mathcal{M}\{\Lambda^c\}.\end{aligned}$$

Example 1.12: Generally speaking, an event Λ is not independent of itself because

$$\mathcal{M}\{\Lambda \cap \Lambda^c\} \neq \mathcal{M}\{\Lambda\} \wedge \mathcal{M}\{\Lambda^c\}$$

whenever $\mathcal{M}\{\Lambda\}$ is neither 1 nor 0.

Theorem 1.7 *(Liu [129]) The events $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ are independent if and only if*

$$\mathcal{M}\left\{\bigcup_{i=1}^{n} \Lambda_i^*\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\Lambda_i^*\} \tag{1.38}$$

where Λ_i^* are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c, \emptyset\}$, $i = 1, 2, \cdots, n$, respectively, and $\emptyset$ is the impossible event.

Proof: Assume $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ are independent events. It follows from the duality of uncertain measure that

$$\mathcal{M}\left\{\bigcup_{i=1}^{n} \Lambda_i^*\right\} = 1 - \mathcal{M}\left\{\bigcap_{i=1}^{n} \Lambda_i^{*c}\right\} = 1 - \bigwedge_{i=1}^{n} \mathcal{M}\{\Lambda_i^{*c}\} = \bigvee_{i=1}^{n} \mathcal{M}\{\Lambda_i^*\}$$

where Λ_i^* are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c, \emptyset\}$, $i = 1, 2, \cdots, n$, respectively. The equation (1.38) is proved. Conversely, if the equation (1.38) holds, then

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} \Lambda_i^*\right\} = 1 - \mathcal{M}\left\{\bigcup_{i=1}^{n} \Lambda_i^{*c}\right\} = 1 - \bigvee_{i=1}^{n} \mathcal{M}\{\Lambda_i^{*c}\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\Lambda_i^*\}.$$

where Λ_i^* are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c, \Gamma\}$, $i = 1, 2, \cdots, n$, respectively. The equation (1.36) is true. The theorem is proved.

Theorem 1.8 *(Liu [137]) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces and $\Lambda_k \in \mathcal{L}_k$ for $k = 1, 2, \cdots, n$. Then the events*

$$\Gamma_1 \times \cdots \times \Gamma_{k-1} \times \Lambda_k \times \Gamma_{k+1} \times \cdots \times \Gamma_n, \quad k = 1, 2, \cdots, n \tag{1.39}$$

are always independent in the product uncertainty space. That is, the events

$$\Lambda_1,\ \Lambda_2,\ \cdots,\Lambda_n \tag{1.40}$$

are always independent if they are from different uncertainty spaces.

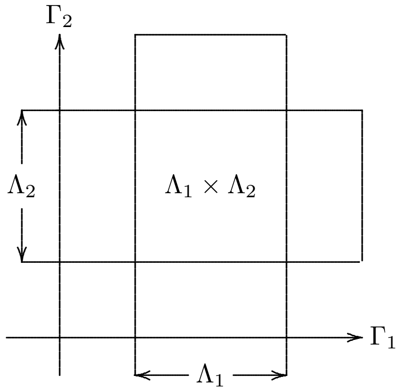

Figure 1.2: $(\Lambda_1 \times \Gamma_2) \cap (\Gamma_1 \times \Lambda_2) = \Lambda_1 \times \Lambda_2$

Proof: For simplicity, we only prove the case of $n = 2$. It follows from the product axiom that the product uncertain measure of the intersection is

$$\mathcal{M}\{(\Lambda_1 \times \Gamma_2) \cap (\Gamma_1 \times \Lambda_2)\} = \mathcal{M}\{\Lambda_1 \times \Lambda_2\} = \mathcal{M}_1\{\Lambda_1\} \wedge \mathcal{M}_2\{\Lambda_2\}.$$

By using $\mathcal{M}\{\Lambda_1 \times \Gamma_2\} = \mathcal{M}_1\{\Lambda_1\}$ and $\mathcal{M}\{\Gamma_1 \times \Lambda_2\} = \mathcal{M}_2\{\Lambda_2\}$, we obtain

$$\mathcal{M}\{(\Lambda_1 \times \Gamma_2) \cap (\Gamma_1 \times \Lambda_2)\} = \mathcal{M}\{\Lambda_1 \times \Gamma_2\} \wedge \mathcal{M}\{\Gamma_1 \times \Lambda_2\}.$$

Similarly, we may prove that

$$\mathcal{M}\{(\Lambda_1 \times \Gamma_2)^c \cap (\Gamma_1 \times \Lambda_2)\} = \mathcal{M}\{(\Lambda_1 \times \Gamma_2)^c\} \wedge \mathcal{M}\{\Gamma_1 \times \Lambda_2\},$$
$$\mathcal{M}\{(\Lambda_1 \times \Gamma_2) \cap (\Gamma_1 \times \Lambda_2)^c\} = \mathcal{M}\{\Lambda_1 \times \Gamma_2\} \wedge \mathcal{M}\{(\Gamma_1 \times \Lambda_2)^c\},$$
$$\mathcal{M}\{(\Lambda_1 \times \Gamma_2)^c \cap (\Gamma_1 \times \Lambda_2)^c\} = \mathcal{M}\{(\Lambda_1 \times \Gamma_2)^c\} \wedge \mathcal{M}\{(\Gamma_1 \times \Lambda_2)^c\}.$$

Thus $\Lambda_1 \times \Gamma_2$ and $\Gamma_1 \times \Lambda_2$ are independent events. Furthermore, since Λ_1 and Λ_2 are understood as $\Lambda_1 \times \Gamma_2$ and $\Gamma_1 \times \Lambda_2$ in the product uncertainty space, respectively, the two events Λ_1 and Λ_2 are also independent.

1.7 Polyrectangular Theorem

Let $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ and $(\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$ be two uncertainty spaces, $\Lambda_1 \in \mathcal{L}_1$ and $\Lambda_2 \in \mathcal{L}_2$. It follows from the product axiom that the rectangle $\Lambda_1 \times \Lambda_2$ has an uncertain measure

$$\mathcal{M}\{\Lambda_1 \times \Lambda_2\} = \mathcal{M}_1\{\Lambda_1\} \wedge \mathcal{M}_2\{\Lambda_2\}. \tag{1.41}$$

This section will extend this result to a more general case.

Definition 1.11 *(Liu [137]) Let* $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ *and* $(\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$ *be two uncertainty spaces. A set on* $\Gamma_1 \times \Gamma_2$ *is called a polyrectangle if it has the form*

$$\Lambda = \bigcup_{i=1}^{m} (\Lambda_{1i} \times \Lambda_{2i}) \tag{1.42}$$

where $\Lambda_{1i} \in \mathcal{L}_1$ *and* $\Lambda_{2i} \in \mathcal{L}_2$ *for* $i = 1, 2, \cdots, m$, *and*

$$\Lambda_{11} \subset \Lambda_{12} \subset \cdots \subset \Lambda_{1m}, \tag{1.43}$$

$$\Lambda_{21} \supset \Lambda_{22} \supset \cdots \supset \Lambda_{2m}. \tag{1.44}$$

A rectangle $\Lambda_1 \times \Lambda_2$ is clearly a polyrectangle. In addition, a "cross"-like set is also a polyrectangle. See Figure 1.3.

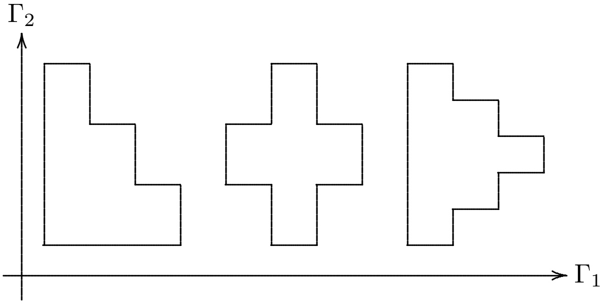

Figure 1.3: Three Polyrectangles

Theorem 1.9 *(Liu [137], Polyrectangular Theorem) Let* $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ *and* $(\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$ *be two uncertainty spaces. Then the polyrectangle*

$$\Lambda = \bigcup_{i=1}^{m} (\Lambda_{1i} \times \Lambda_{2i}) \tag{1.45}$$

on the product uncertainty space $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1) \times (\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$ *has an uncertain measure*

$$\mathcal{M}\{\Lambda\} = \bigvee_{i=1}^{m} \mathcal{M}_1\{\Lambda_{1i}\} \wedge \mathcal{M}_2\{\Lambda_{2i}\}. \tag{1.46}$$

Proof: It is clear that the maximum rectangle in the polyrectangle Λ is one of $\Lambda_{1i} \times \Lambda_{2i}$, $i = 1, 2, \cdots, n$. Denote the maximum rectangle by $\Lambda_{1k} \times \Lambda_{2k}$. Case I: If

$$\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} = \mathcal{M}_1\{\Lambda_{1k}\},$$

then the maximum rectangle in Λ^c is $\Lambda_{1k}^c \times \Lambda_{2,k+1}^c$, and

$$\mathcal{M}\{\Lambda_{1k}^c \times \Lambda_{2,k+1}^c\} = \mathcal{M}_1\{\Lambda_{1k}^c\} = 1 - \mathcal{M}_1\{\Lambda_{1k}\}.$$

Thus

$$\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} + \mathcal{M}\{\Lambda_{1k}^c \times \Lambda_{2,k+1}^c\} = 1.$$

Case II: If

$$\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} = \mathcal{M}_2\{\Lambda_{2k}\},$$

then the maximum rectangle in Λ^c is $\Lambda_{1,k-1}^c \times \Lambda_{2k}^c$, and

$$\mathcal{M}\{\Lambda_{1,k-1}^c \times \Lambda_{2k}^c\} = \mathcal{M}_2\{\Lambda_{2k}^c\} = 1 - \mathcal{M}_2\{\Lambda_{2k}\}.$$

Thus

$$\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} + \mathcal{M}\{\Lambda_{1,k-1}^c \times \Lambda_{2k}^c\} = 1.$$

No matter what case happens, the sum of the uncertain measures of the maximum rectangles in Λ and Λ^c is always 1. It follows from the product axiom that (1.46) holds.

Remark 1.10: Note that the polyrectangular theorem is also applicable to the polyrectangles that are unions of infinitely many rectangles. In this case, the polyrectangles may become the shapes in Figure 1.4.

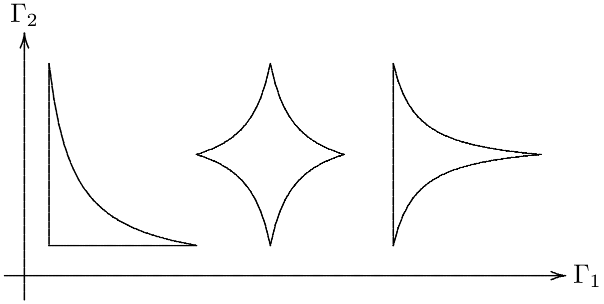

Figure 1.4: Three Deformed Polyrectangles

1.8 Conditional Uncertain Measure

We consider the uncertain measure of an event A after it has been learned that some other event B has occurred. This new uncertain measure of A is called the *conditional uncertain measure* of A given B.

In order to define a conditional uncertain measure $\mathcal{M}\{A|B\}$, at first we have to enlarge $\mathcal{M}\{A \cap B\}$ because $\mathcal{M}\{A \cap B\} < 1$ for all events whenever $\mathcal{M}\{B\} < 1$. It seems that we have no alternative but to divide $\mathcal{M}\{A \cap B\}$ by $\mathcal{M}\{B\}$. Unfortunately, $\mathcal{M}\{A \cap B\}/\mathcal{M}\{B\}$ is not always an uncertain measure. However, the value $\mathcal{M}\{A|B\}$ should not be greater than $\mathcal{M}\{A \cap B\}/\mathcal{M}\{B\}$ (otherwise the normality will be lost), i.e.,

$$\mathcal{M}\{A|B\} \le \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}}. \tag{1.47}$$

On the other hand, in order to preserve the duality, we should have

$$\mathcal{M}\{A|B\} = 1 - \mathcal{M}\{A^c|B\} \geq 1 - \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}}. \tag{1.48}$$

Furthermore, since $(A \cap B) \cup (A^c \cap B) = B$, we have $\mathcal{M}\{B\} \leq \mathcal{M}\{A \cap B\} + \mathcal{M}\{A^c \cap B\}$ by using the subadditivity axiom. Thus

$$0 \leq 1 - \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} \leq \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} \leq 1. \tag{1.49}$$

Hence any numbers between $1-\mathcal{M}\{A^c \cap B\}/\mathcal{M}\{B\}$ and $\mathcal{M}\{A \cap B\}/\mathcal{M}\{B\}$ are reasonable values that the conditional uncertain measure may take. Based on the maximum uncertainty principle (Liu [122]), we have the following conditional uncertain measure.

Definition 1.12 *(Liu [122]) Let* $(\Gamma, \mathcal{L}, \mathcal{M})$ *be an uncertainty space, and* $A, B \in \mathcal{L}$*. Then the conditional uncertain measure of* A *given* B *is defined by*

$$\mathcal{M}\{A|B\} = \begin{cases} \dfrac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}}, & \text{if } \dfrac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} < 0.5 \\ 1 - \dfrac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}}, & \text{if } \dfrac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} < 0.5 \\ 0.5, & \text{otherwise} \end{cases} \tag{1.50}$$

provided that $\mathcal{M}\{B\} > 0$*.*

Remark 1.11: It follows immediately from the definition of conditional uncertain measure that

$$1 - \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} \leq \mathcal{M}\{A|B\} \leq \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}}. \tag{1.51}$$

Furthermore, the conditional uncertain measure obeys the maximum uncertainty principle, and takes values as close to 0.5 as possible.

Remark 1.12: The conditional uncertain measure $\mathcal{M}\{A|B\}$ yields the posterior uncertain measure of A after the occurrence of event B.

Theorem 1.10 *Let* $(\Gamma, \mathcal{L}, \mathcal{M})$ *be an uncertainty space, and let* B *be an event with* $\mathcal{M}\{B\} > 0$*. Then* $\mathcal{M}\{\cdot|B\}$ *defined by (1.50) is an uncertain measure, and* $(\Gamma, \mathcal{L}, \mathcal{M}\{\cdot|B\})$ *is an uncertainty space.*

Proof: It is sufficient to prove that $\mathcal{M}\{\cdot|B\}$ satisfies the normality, duality and subadditivity axioms. At first, it satisfies the normality axiom, i.e.,

$$\mathcal{M}\{\Gamma|B\} = 1 - \frac{\mathcal{M}\{\Gamma^c \cap B\}}{\mathcal{M}\{B\}} = 1 - \frac{\mathcal{M}\{\emptyset\}}{\mathcal{M}\{B\}} = 1.$$

For any event A, if

$$\frac{\mathcal{M}\{A\cap B\}}{\mathcal{M}\{B\}}\ge 0.5,\quad \frac{\mathcal{M}\{A^c\cap B\}}{\mathcal{M}\{B\}}\ge 0.5,$$

then we have $\mathcal{M}\{A|B\}+\mathcal{M}\{A^c|B\}=0.5+0.5=1$ immediately. Otherwise, without loss of generality, suppose

$$\frac{\mathcal{M}\{A\cap B\}}{\mathcal{M}\{B\}}<0.5<\frac{\mathcal{M}\{A^c\cap B\}}{\mathcal{M}\{B\}},$$

then we have

$$\mathcal{M}\{A|B\}+\mathcal{M}\{A^c|B\}=\frac{\mathcal{M}\{A\cap B\}}{\mathcal{M}\{B\}}+\left(1-\frac{\mathcal{M}\{A\cap B\}}{\mathcal{M}\{B\}}\right)=1.$$

That is, $\mathcal{M}\{\cdot|B\}$ satisfies the duality axiom. Finally, for any countable sequence $\{A_i\}$ of events, if $\mathcal{M}\{A_i|B\}<0.5$ for all i, it follows from (1.51) and the subadditivity axiom that

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}A_i\,|\,B\right\}\le\frac{\mathcal{M}\left\{\bigcup_{i=1}^{\infty}A_i\cap B\right\}}{\mathcal{M}\{B\}}\le\frac{\sum_{i=1}^{\infty}\mathcal{M}\{A_i\cap B\}}{\mathcal{M}\{B\}}=\sum_{i=1}^{\infty}\mathcal{M}\{A_i|B\}.$$

Suppose there is one term greater than 0.5, say

$$\mathcal{M}\{A_1|B\}\ge 0.5,\quad \mathcal{M}\{A_i|B\}<0.5,\quad i=2,3,\cdots$$

If $\mathcal{M}\{\cup_i A_i|B\}=0.5$, then we immediately have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}A_i\,|\,B\right\}\le\sum_{i=1}^{\infty}\mathcal{M}\{A_i|B\}.$$

If $\mathcal{M}\{\cup_i A_i|B\}>0.5$, we may prove the above inequality by the following facts:

$$A_1^c\cap B\subset\bigcup_{i=2}^{\infty}(A_i\cap B)\cup\left(\bigcap_{i=1}^{\infty}A_i^c\cap B\right),$$

$$\mathcal{M}\{A_1^c\cap B\}\le\sum_{i=2}^{\infty}\mathcal{M}\{A_i\cap B\}+\mathcal{M}\left\{\bigcap_{i=1}^{\infty}A_i^c\cap B\right\},$$

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}A_i\,|\,B\right\}=1-\frac{\mathcal{M}\left\{\bigcap_{i=1}^{\infty}A_i^c\cap B\right\}}{\mathcal{M}\{B\}},$$

$$\sum_{i=1}^{\infty}\mathcal{M}\{A_i|B\} \geq 1 - \frac{\mathcal{M}\{A_1^c \cap B\}}{\mathcal{M}\{B\}} + \frac{\sum_{i=2}^{\infty}\mathcal{M}\{A_i \cap B\}}{\mathcal{M}\{B\}}.$$

If there are at least two terms greater than 0.5, then the subadditivity is clearly true. Thus $\mathcal{M}\{\cdot|B\}$ satisfies the subadditivity axiom. Hence $\mathcal{M}\{\cdot|B\}$ is an uncertain measure. Furthermore, $(\Gamma, \mathcal{L}, \mathcal{M}\{\cdot|B\})$ is an uncertainty space.

1.9 Bibliographic Notes

When no samples are available to estimate a probability distribution, we have to invite some domain experts to evaluate the belief degree that each event will happen. Perhaps some people think that the belief degree is subjective probability or fuzzy concept. However, Liu [131] declared that it is usually inappropriate because both probability theory and fuzzy set theory may lead to counterintuitive results in this case.

In order to rationally deal with belief degrees, uncertainty theory was founded by Liu [122] in 2007 and perfected by Liu [125] in 2009 with the normality axiom, duality axiom, subadditivity axiom, and product axiom of uncertain measure.

Furthermore, uncertain measure was also actively investigated by Gao [48], Liu [129], Zhang [268], Peng and Iwamura [185], and Liu [137], among others. Since then, the tool of uncertain measure was well developed and became a rigorous footstone of uncertainty theory.

Chapter 2

Uncertain Variable

Uncertain variable is a fundamental concept in uncertainty theory. It is used to represent quantities with uncertainty. The emphasis in this chapter is mainly on uncertain variable, uncertainty distribution, independence, operational law, expected value, variance, moments, entropy, distance, conditional uncertainty distribution, uncertain sequence, and uncertain vector.

2.1 Uncertain Variable

Roughly speaking, an uncertain variable is a measurable function on an uncertainty space. A formal definition is given as follows.

Definition 2.1 *(Liu [122]) An uncertain variable is a function* ξ *from an uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *to the set of real numbers such that* $\{\xi \in B\}$ *is an event for any Borel set* B.

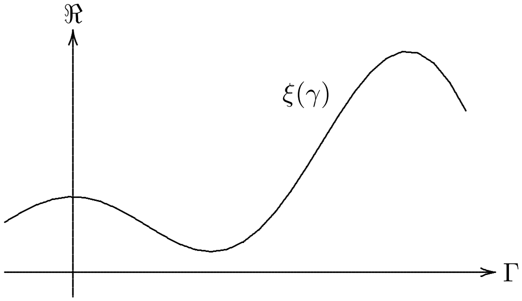

Figure 2.1: An Uncertain Variable. Reprinted from Liu [129].

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5_3

Example 2.1: Take $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 0.5$. Then the function

$$\xi(\gamma) = \begin{cases} 0, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2 \end{cases}$$

is an uncertain variable.

Example 2.2: A crisp number b may be regarded as a special uncertain variable. In fact, it is the constant function $\xi(\gamma) \equiv b$ on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$.

Definition 2.2 *An uncertain variable ξ on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be (a) nonnegative if $\mathcal{M}\{\xi < 0\} = 0$; and (b) positive if $\mathcal{M}\{\xi \le 0\} = 0$.*

Definition 2.3 *Let ξ and η be uncertain variables defined on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. We say $\xi = \eta$ if $\xi(\gamma) = \eta(\gamma)$ for almost all $\gamma \in \Gamma$.*

Definition 2.4 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain variables, and let f be a real-valued measurable function. Then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable defined by*

$$\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \cdots, \xi_n(\gamma)), \quad \forall \gamma \in \Gamma. \tag{2.1}$$

Example 2.3: Let ξ_1 and ξ_2 be two uncertain variables. Then the sum $\xi = \xi_1 + \xi_2$ is an uncertain variable defined by

$$\xi(\gamma) = \xi_1(\gamma) + \xi_2(\gamma), \quad \forall \gamma \in \Gamma.$$

The product $\xi = \xi_1 \xi_2$ is also an uncertain variable defined by

$$\xi(\gamma) = \xi_1(\gamma) \cdot \xi_2(\gamma), \quad \forall \gamma \in \Gamma.$$

The reader may wonder whether $\xi(\gamma)$ defined by (2.1) is an uncertain variable. The following theorem answers this question.

Theorem 2.1 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain variables, and let f be a real-valued measurable function. Then $f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable.*

Proof: Since $\xi_1, \xi_2, \cdots, \xi_n$ are uncertain variables, they are measurable functions from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers. Thus $f(\xi_1, \xi_2, \cdots, \xi_n)$ is also a measurable function from the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers. Hence $f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain variable.

2.2 Uncertainty Distribution

This section introduces a concept of uncertainty distribution in order to describe uncertain variables. Mention that uncertainty distribution is a carrier of incomplete information of uncertain variable. However, in many cases, it is sufficient to know the uncertainty distribution rather than the uncertain variable itself.

Definition 2.5 *(Liu [122]) The uncertainty distribution Φ of an uncertain variable ξ is defined by*

$$\Phi(x) = \mathcal{M}\{\xi \le x\} \tag{2.2}$$

for any real number x.

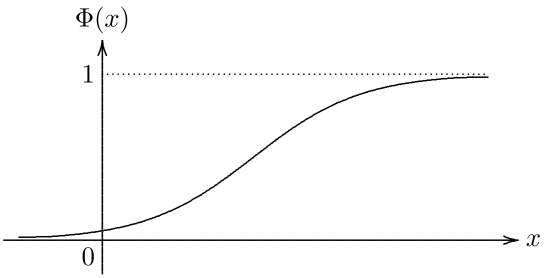

Figure 2.2: An Uncertainty Distribution. Reprinted from Liu [129].

Exercise 2.1: A real number b is a special uncertain variable $\xi(\gamma) \equiv b$. Show that such an uncertain variable has an uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < b \\ 1, & \text{if } x \ge b. \end{cases}$$

Exercise 2.2: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}\{\gamma_1\} = 0.7$ and $\mathcal{M}\{\gamma_2\} = 0.3$. Show that the uncertain variable

$$\xi(\gamma) = \begin{cases} 0, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2 \end{cases}$$

has an uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 0 \\ 0.7, & \text{if } 0 \le x < 1 \\ 1, & \text{if } 1 \le x. \end{cases}$$

Exercise 2.3: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3\}$ with

$$\mathcal{M}\{\gamma_1\} = 0.6, \quad \mathcal{M}\{\gamma_2\} = 0.3, \quad \mathcal{M}\{\gamma_3\} = 0.2.$$

Show that the uncertain variable

$$\xi(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ 2, & \text{if } \gamma = \gamma_2 \\ 3, & \text{if } \gamma = \gamma_3 \end{cases}$$

has an uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 1 \\ 0.6, & \text{if } 1 \le x < 2 \\ 0.8, & \text{if } 2 \le x < 3 \\ 1, & \text{if } 3 \le x. \end{cases}$$

Exercise 2.4: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be the interval $[0, 1]$ with Borel algebra and Lebesgue measure. Show that the uncertain variable $\xi(\gamma) = \gamma^2$ has an uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 0 \\ \sqrt{x}, & \text{if } 0 \le x \le 1 \\ 1, & \text{if } x \ge 1. \end{cases} \tag{2.3}$$

Definition 2.6 *Uncertain variables are said to be identically distributed if they have the same uncertainty distribution.*

It is clear that uncertain variables ξ and η are identically distributed if $\xi = \eta$. However, identical distribution does not imply $\xi = \eta$. For example, let $(\Gamma, \mathcal{L}, \mathcal{M})$ be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 0.5$. Define

$$\xi(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ -1, & \text{if } \gamma = \gamma_2, \end{cases} \qquad \eta(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2. \end{cases}$$

Then ξ and η have the same uncertainty distribution,

$$\Phi(x) = \begin{cases} 0, & \text{if } x < -1 \\ 0.5, & \text{if } -1 \le x < 1 \\ 1, & \text{if } x \ge 1. \end{cases}$$

Thus the two uncertain variables ξ and η are identically distributed but $\xi \ne \eta$.

Sufficient and Necessary Condition

Theorem 2.2 *(Peng-Iwamura Theorem [184]) A function* $\Phi(x) : \Re \to [0,1]$ *is an uncertainty distribution if and only if it is a monotone increasing function except* $\Phi(x) \equiv 0$ *and* $\Phi(x) \equiv 1$*.*

Proof: It is obvious that an uncertainty distribution Φ is a monotone increasing function. In addition, both $\Phi(x) \not\equiv 0$ and $\Phi(x) \not\equiv 1$ follow from the asymptotic theorem immediately. Conversely, suppose that Φ is a monotone increasing function but $\Phi(x) \not\equiv 0$ and $\Phi(x) \not\equiv 1$. We will prove that there is an uncertain variable whose uncertainty distribution is just Φ. Let $\mathcal{C}$ be a collection of all intervals of the form $(-\infty, a]$, (b, ∞), $\emptyset$ and $\Re$. We define a set function on $\Re$ as follows,

$$\begin{gathered}\mathcal{M}\{(-\infty, a]\} = \Phi(a), \\ \mathcal{M}\{(b, +\infty)\} = 1 - \Phi(b), \\ \mathcal{M}\{\emptyset\} = 0, \quad \mathcal{M}\{\Re\} = 1.\end{gathered}$$

For an arbitrary Borel set B, there exists a sequence $\{A_i\}$ in $\mathcal{C}$ such that

$$B \subset \bigcup_{i=1}^{\infty} A_i.$$

Note that such a sequence is not unique. Thus the set function $\mathcal{M}\{B\}$ is defined by

$$\mathcal{M}\{B\} = \begin{cases} \inf\limits_{B \subset \bigcup_{i=1}^{\infty} A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\}, & \text{if } \inf\limits_{B \subset \bigcup_{i=1}^{\infty} A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5 \\ 1 - \inf\limits_{B^c \subset \bigcup_{i=1}^{\infty} A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\}, & \text{if } \inf\limits_{B^c \subset \bigcup_{i=1}^{\infty} A_i} \sum\limits_{i=1}^{\infty} \mathcal{M}\{A_i\} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

We may prove that the set function $\mathcal{M}$ is indeed an uncertain measure on $\Re$, and the uncertain variable defined by the identity function $\xi(\gamma) = \gamma$ from the uncertainty space $(\Re, \mathcal{L}, \mathcal{M})$ to $\Re$ has the uncertainty distribution Φ.

Example 2.4: Let c be a number with $0 < c < 1$. Then $\Phi(x) \equiv c$ is an uncertainty distribution. When $c \le 0.5$, we define a set function over $\Re$ as follows,

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ c, & \text{if } \Lambda \text{ is upper bounded} \\ 0.5, & \text{if both } \Lambda \text{ and } \Lambda^c \text{ are upper unbounded} \\ 1 - c, & \text{if } \Lambda^c \text{ is upper bounded} \\ 1, & \text{if } \Lambda = \Gamma. \end{cases}$$

Then $(\Re, \mathcal{L}, \mathcal{M})$ is an uncertainty space. It is easy to verify that the identity function $\xi(\gamma) = \gamma$ is an uncertain variable whose uncertainty distribution is just $\Phi(x) \equiv c$. When $c > 0.5$, we define

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ 1 - c, & \text{if } \Lambda \text{ is upper bounded} \\ 0.5, & \text{if both } \Lambda \text{ and } \Lambda^c \text{ are upper unbounded} \\ c, & \text{if } \Lambda^c \text{ is upper bounded} \\ 1, & \text{if } \Lambda = \Gamma. \end{cases}$$

Then the function $\xi(\gamma) = -\gamma$ is an uncertain variable whose uncertainty distribution is just $\Phi(x) \equiv c$.

What is a "completely unknown number"?

A "completely unknown number" may be regarded as an uncertain variable whose uncertainty distribution is

$$\Phi(x) \equiv 0.5 \tag{2.4}$$

for any real number x.

What is a "large number"?

A "large number" may be regarded as an uncertain variable. A possible uncertainty distribution is

$$\Phi(x) = \frac{1}{2}\left(1 + \exp(1000 - x)\right)^{-1} \tag{2.5}$$

for any real number x.

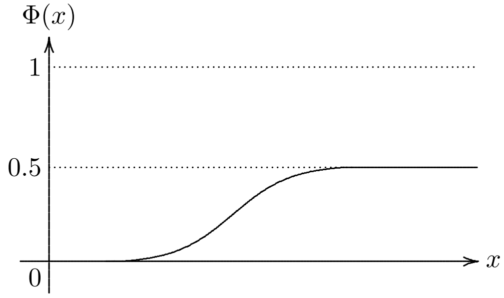

Figure 2.3: Uncertainty Distribution of "Large Number"

What is a "small number"?

A "small number" may be regarded as an uncertain variable. A possible uncertainty distribution is

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le 0 \\ (1+\exp(-10x))^{-1}, & \text{if } x > 0. \end{cases} \tag{2.6}$$

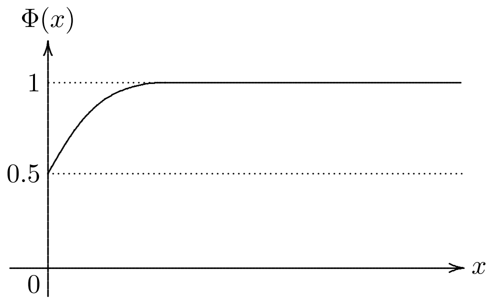

Figure 2.4: Uncertainty Distribution of "Small Number"

How old is John?

Someone thinks John is neither younger than 24 nor older than 28, and presents an uncertainty distribution of John's age as follows,

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le 24 \\ (x-24)/4, & \text{if } 24 \le x \le 28 \\ 1, & \text{if } x \ge 28. \end{cases} \tag{2.7}$$

How tall is James?

Someone thinks James' height is between 180 and 185 centimeters, and presents the following uncertainty distribution,

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le 180 \\ (x-180)/5, & \text{if } 180 \le x \le 185 \\ 1, & \text{if } x \ge 185. \end{cases} \tag{2.8}$$

Some Uncertainty Distributions

Definition 2.7 *An uncertain variable ξ is called linear if it has a linear uncertainty distribution*

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a \\ (x-a)/(b-a), & \text{if } a \le x \le b \\ 1, & \text{if } x \ge b \end{cases} \tag{2.9}$$

denoted by $\mathcal{L}(a,b)$ where a and b are real numbers with $a < b$.

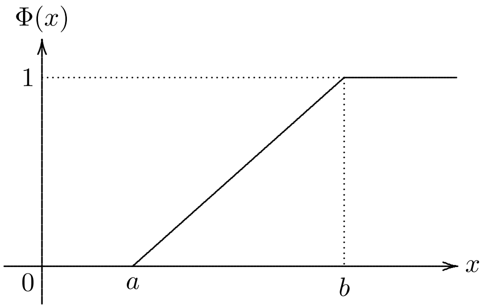

Figure 2.5: Linear Uncertainty Distribution. Reprinted from Liu [129].

Example 2.5: John's age (2.7) is a linear uncertain variable $\mathcal{L}(24,28)$, and James' height (2.8) is another linear uncertain variable $\mathcal{L}(180,185)$.

Definition 2.8 *An uncertain variable ξ is called zigzag if it has a zigzag uncertainty distribution*

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le a \\ (x-a)/2(b-a), & \text{if } a \le x \le b \\ (x+c-2b)/2(c-b), & \text{if } b \le x \le c \\ 1, & \text{if } x \ge c \end{cases} \tag{2.10}$$

denoted by $\mathcal{Z}(a,b,c)$ where a,b,c are real numbers with $a < b < c$.

Definition 2.9 *An uncertain variable ξ is called normal if it has a normal uncertainty distribution*

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \Re \tag{2.11}$$

denoted by $\mathcal{N}(e,\sigma)$ where e and σ are real numbers with $\sigma > 0$.

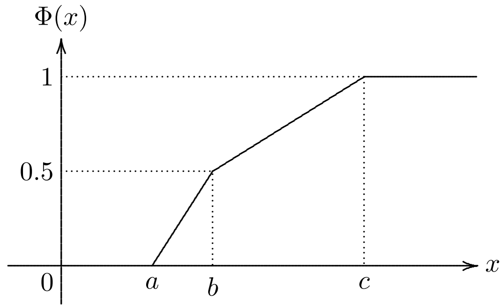

Figure 2.6: Zigzag Uncertainty Distribution. Reprinted from Liu [129].

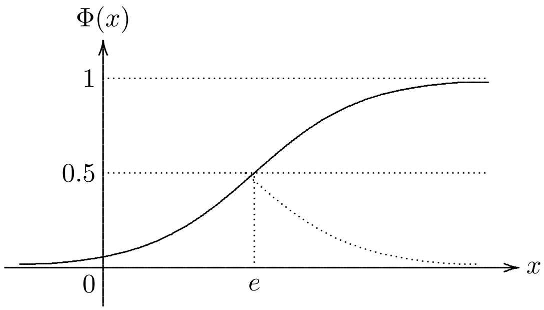

Figure 2.7: Normal Uncertainty Distribution. Reprinted from Liu [129].

Definition 2.10 *An uncertain variable ξ is called lognormal if $\ln\xi$ is a normal uncertain variable $\mathcal{N}(e,\sigma)$. In other words, a lognormal uncertain variable has an uncertainty distribution*

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e - \ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \ge 0 \tag{2.12}$$

denoted by $\mathcal{LOGN}(e,\sigma)$, where e and σ are real numbers with $\sigma > 0$.

Definition 2.11 *An uncertain variable ξ is called empirical if it has an empirical uncertainty distribution*

$$\Phi(x) = \begin{cases} 0, & \text{if } x < x_1 \\ \alpha_i + \dfrac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i}, & \text{if } x_i \le x \le x_{i+1},\ 1 \le i < n \\ 1, & \text{if } x > x_n \end{cases} \tag{2.13}$$

where $x_1 < x_2 < \cdots < x_n$ and $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le 1$.

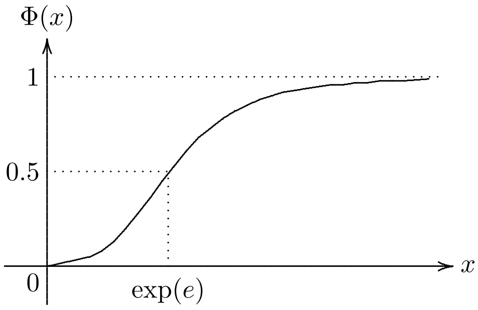

Figure 2.8: Lognormal Uncertainty Distribution. Reprinted from Liu [129].

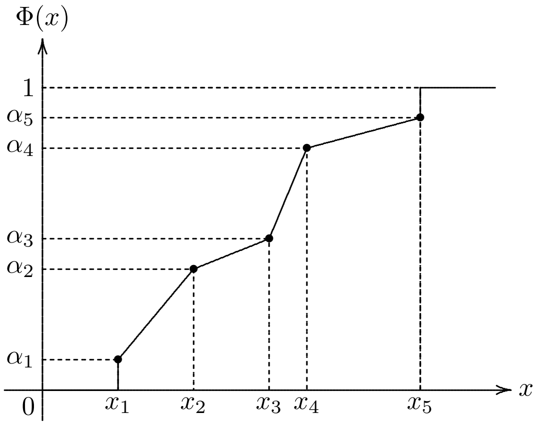

Figure 2.9: Empirical Uncertainty Distribution

Measure Inversion Theorem

Theorem 2.3 *(Liu [129], Measure Inversion Theorem) Let ξ be an uncertain variable with uncertainty distribution Φ. Then for any real number x, we have*

$$\mathcal{M}\{\xi \le x\} = \Phi(x), \quad \mathcal{M}\{\xi > x\} = 1 - \Phi(x). \tag{2.14}$$

Proof: The equation $\mathcal{M}\{\xi \le x\} = \Phi(x)$ follows from the definition of uncertainty distribution immediately. By using the duality of uncertain measure, we get

$$\mathcal{M}\{\xi > x\} = 1 - \mathcal{M}\{\xi \le x\} = 1 - \Phi(x).$$

The theorem is verified.

Remark 2.1: When the uncertainty distribution Φ is a continuous function, we also have

$$\mathcal{M}\{\xi < x\} = \Phi(x), \quad \mathcal{M}\{\xi \ge x\} = 1 - \Phi(x). \tag{2.15}$$

Theorem 2.4 *Let ξ be an uncertain variable with continuous uncertainty distribution Φ. Then for any interval $[a,b]$, we have*

$$\Phi(b) - \Phi(a) \le \mathcal{M}\{a \le \xi \le b\} \le \Phi(b) \wedge (1 - \Phi(a)). \tag{2.16}$$

Proof: It follows from the subadditivity of uncertain measure and the measure inversion theorem that

$$\mathcal{M}\{a \le \xi \le b\} + \mathcal{M}\{\xi \le a\} \ge \mathcal{M}\{\xi \le b\}.$$

That is,

$$\mathcal{M}\{a \le \xi \le b\} + \Phi(a) \ge \Phi(b).$$

Thus the inequality on the left hand side is verified. It follows from the monotonicity of uncertain measure and the measure inversion theorem that

$$\mathcal{M}\{a \le \xi \le b\} \le \mathcal{M}\{\xi \in (-\infty, b]\} = \Phi(b).$$

On the other hand,

$$\mathcal{M}\{a \le x \le b\} \le \mathcal{M}\{\xi \in [a, +\infty)\} = 1 - \Phi(a).$$

Hence the inequality on the right hand side is proved.

Remark 2.2: Perhaps some readers would like to get an exactly scalar value of the uncertain measure $\mathcal{M}\{a \le x \le b\}$. Generally speaking, it is an impossible job (except $a = -\infty$ or $b = +\infty$) if only an uncertainty distribution is available. I would like to ask if there is a need to know it. In fact, it is not necessary for practical purpose. Would you believe? I hope so!

Regular Uncertainty Distribution

Definition 2.12 *(Liu [129]) An uncertainty distribution $\Phi(x)$ is said to be regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and*

$$\lim_{x \to -\infty} \Phi(x) = 0, \qquad \lim_{x \to +\infty} \Phi(x) = 1. \tag{2.17}$$

For example, linear uncertainty distribution, zigzag uncertainty distribution, normal uncertainty distribution, and lognormal uncertainty distribution are all regular.

Stipulation 2.1 *The uncertainty distribution of a crisp value c is regular. That is, we will say*

$$\Phi(x) = \begin{cases} 1, & \text{if } x \ge c \\ 0, & \text{if } x < c \end{cases} \tag{2.18}$$

is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$ even though it is discontinuous at c.

Inverse Uncertainty Distribution

It is clear that a regular uncertainty distribution $\Phi(x)$ has an inverse function on the range of x with $0 < \Phi(x) < 1$, and the inverse function $\Phi^{-1}(\alpha)$ exists on the open interval $(0, 1)$.

Definition 2.13 *(Liu [129]) Let ξ be an uncertain variable with regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ is called the inverse uncertainty distribution of ξ.*

Note that the inverse uncertainty distribution $\Phi^{-1}(\alpha)$ is well defined on the open interval $(0, 1)$. If needed, we may extend the domain to $[0, 1]$ via

$$\Phi^{-1}(0) = \lim_{\alpha\downarrow 0} \Phi^{-1}(\alpha), \quad \Phi^{-1}(1) = \lim_{\alpha\uparrow 1} \Phi^{-1}(\alpha). \tag{2.19}$$

Example 2.6: The inverse uncertainty distribution of linear uncertain variable $\mathcal{L}(a, b)$ is

$$\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b. \tag{2.20}$$

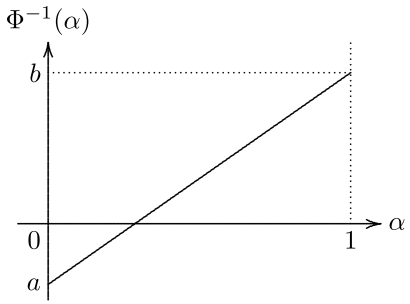

Figure 2.10: Inverse Linear Uncertainty Distribution. Reprinted from Liu [129].

Example 2.7: The inverse uncertainty distribution of zigzag uncertain variable $\mathcal{Z}(a, b, c)$ is

$$\Phi^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)a + 2\alpha b, & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)b + (2\alpha - 1)c, & \text{if } \alpha \geq 0.5. \end{cases} \tag{2.21}$$

Example 2.8: The inverse uncertainty distribution of normal uncertain variable $\mathcal{N}(e, \sigma)$ is

$$\Phi^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}. \tag{2.22}$$

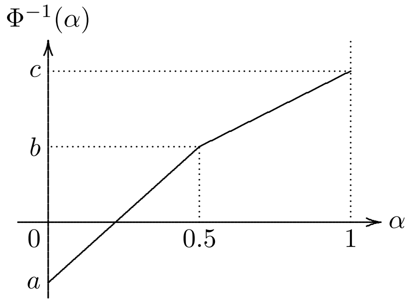

Figure 2.11: Inverse Zigzag Uncertainty Distribution. Reprinted from Liu [129].

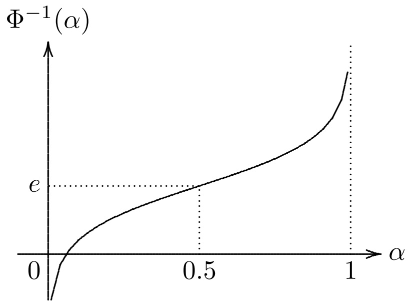

Figure 2.12: Inverse Normal Uncertainty Distribution. Reprinted from Liu [129].

Example 2.9: The inverse uncertainty distribution of lognormal uncertain variable $\mathcal{LOGN}(e, \sigma)$ is

$$\Phi^{-1}(\alpha) = \exp\left(e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right). \tag{2.23}$$

Theorem 2.5 *A function Φ^{-1} is an inverse uncertainty distribution of an uncertain variable ξ if and only if*

$$\mathcal{M}\{\xi \le \Phi^{-1}(\alpha)\} = \alpha \tag{2.24}$$

for all $\alpha \in [0, 1]$.

Proof: Suppose Φ^{-1} is the inverse uncertainty distribution of ξ. Then for any α, we have

$$\mathcal{M}\{\xi \le \Phi^{-1}(\alpha)\} = \Phi(\Phi^{-1}(\alpha)) = \alpha.$$

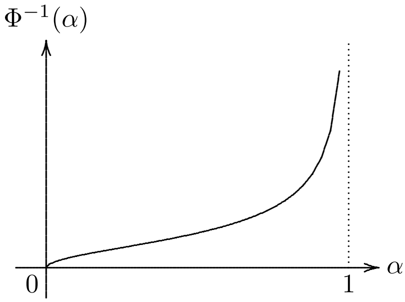

Figure 2.13: Inverse Lognormal Uncertainty Distribution. Reprinted from Liu [129].

Conversely, suppose Φ^{-1} meets (2.24). Write $x = \Phi^{-1}(\alpha)$. Then $\alpha = \Phi(x)$ and

$$\mathcal{M}\{\xi \le x\} = \alpha = \Phi(x).$$

That is, Φ is the uncertainty distribution of ξ and Φ^{-1} is its inverse uncertainty distribution. The theorem is verified.

Theorem 2.6 *(Liu [134], Sufficient and Necessary Condition) A function $\Phi^{-1}(\alpha) : (0,1) \to \Re$ is an inverse uncertainty distribution if and only if it is a continuous and strictly increasing function with respect to α.*

Proof: Suppose $\Phi^{-1}(\alpha)$ is an inverse uncertainty distribution. It follows from the definition of inverse uncertainty distribution that $\Phi^{-1}(\alpha)$ is a continuous and strictly increasing function with respect to $\alpha \in (0,1)$.

Conversely, suppose $\Phi^{-1}(\alpha)$ is a continuous and strictly increasing function on $(0,1)$. Define

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le \lim\limits_{\alpha \downarrow 0} \Phi^{-1}(\alpha) \\ \alpha, & \text{if } x = \Phi^{-1}(\alpha) \\ 1, & \text{if } x \ge \lim\limits_{\alpha \uparrow 1} \Phi^{-1}(\alpha). \end{cases}$$

It follows from Peng-Iwamura theorem that $\Phi(x)$ is an uncertainty distribution of some uncertain variable ξ. Then for each $\alpha \in (0,1)$, we have

$$\mathcal{M}\{\xi \le \Phi^{-1}(\alpha)\} = \Phi(\Phi^{-1}(\alpha)) = \alpha.$$

Thus $\Phi^{-1}(\alpha)$ is just the inverse uncertainty distribution of the uncertain variable ξ. The theorem is verified.

Stipulation 2.2 *We say a crisp value c has an inverse uncertainty distribution*

$$\Phi^{-1}(\alpha) \equiv c \tag{2.25}$$

and $\Phi^{-1}(\alpha)$ *is a continuous and strictly increasing function with respect to* $\alpha \in (0,1)$ *even though it is not.*

2.3 Independence

Independence has been explained in many ways. Personally, I think some uncertain variables are independent if they can be separately defined on different uncertainty spaces. In order to ensure that we are able to do so, we may define independence in the following mathematical form.

Definition 2.14 *(Liu [125]) The uncertain variables* $\xi_1, \xi_2, \cdots, \xi_n$ *are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\} \tag{2.26}$$

for any Borel sets $B_1, B_2, \cdots, B_n$.

Example 2.10: Let $\xi_1(\gamma_1)$ and $\xi_2(\gamma_2)$ be uncertain variables on the uncertainty spaces $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ and $(\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$, respectively. It is clear that they are also uncertain variables on the product uncertainty space $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1) \times (\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$. Then for any Borel sets B_1 and B_2, we have

$$\begin{aligned}
&\mathcal{M}\{(\xi_1 \in B_1) \cap (\xi_2 \in B_2)\} \\
&= \mathcal{M}\{(\gamma_1, \gamma_2) \,|\, \xi_1(\gamma_1) \in B_1,\ \xi_2(\gamma_2) \in B_2\} \\
&= \mathcal{M}\{(\gamma_1 \,|\, \xi_1(\gamma_1) \in B_1) \times (\gamma_2 \,|\, \xi_2(\gamma_2) \in B_2)\} \\
&= \mathcal{M}_1\{\gamma_1 \,|\, \xi_1(\gamma_1) \in B_1\} \wedge \mathcal{M}_2\{\gamma_2 \,|\, \xi_2(\gamma_2) \in B_2\} \\
&= \mathcal{M}\{\xi_1 \in B_1\} \wedge \mathcal{M}\{\xi_2 \in B_2\}.
\end{aligned}$$

Thus ξ_1 and ξ_2 are independent in the product uncertainty space. In fact, it is true that uncertain variables are always independent if they are defined on different uncertainty spaces.

Theorem 2.7 *The uncertain variables* $\xi_1, \xi_2, \cdots, \xi_n$ *are independent if and only if*

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\} \tag{2.27}$$

for any Borel sets $B_1, B_2, \cdots, B_n$.

Proof: It follows from the duality of uncertain measure that $\xi_1, \xi_2, \cdots, \xi_n$ are independent if and only if

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i \in B_i)\right\} = 1 - \mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i^c)\right\}$$
$$= 1 - \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i^c\} = \bigvee_{i=1}^{n} \mathcal{M}\left\{\xi_i \in B_i\right\}.$$

Thus the proof is complete.

Theorem 2.8 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables, and let f_1, $f_2, \cdots, f_n$ be measurable functions. Then $f_1(\xi_1), f_2(\xi_2), \cdots, f_n(\xi_n)$ are independent uncertain variables.*

Proof: For any Borel sets $B_1, B_2, \cdots, B_n$, it follows from the definition of independence that

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(f_i(\xi_i) \in B_i)\right\} = \mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in f_i^{-1}(B_i))\right\}$$
$$= \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in f_i^{-1}(B_i)\} = \bigwedge_{i=1}^{n} \mathcal{M}\{f_i(\xi_i) \in B_i\}.$$

Thus $f_1(\xi_1), f_2(\xi_2), \cdots, f_n(\xi_n)$ are independent uncertain variables.

Example 2.11: Let ξ_1 and ξ_2 be independent uncertain variables. Then their functions $\xi_1 + 2$ and $\xi_2^2 + 3\xi_2 + 4$ are also independent.

2.4 Operational Law

The operational law of independent uncertain variables was given by Liu [129] for calculating the uncertainty distribution of strictly increasing function, strictly decreasing function, and strictly monotone function of uncertain variables. This section will also discuss the uncertainty distribution of Boolean function of Boolean uncertain variables.

Strictly Increasing Function of Uncertain Variables

A real-valued function $f(x_1, x_2, \cdots, x_n)$ is said to be strictly increasing if

$$f(x_1, x_2, \cdots, x_n) \le f(y_1, y_2, \cdots, y_n) \tag{2.28}$$

whenever $x_i \le y_i$ for $i = 1, 2, \cdots, n$, and

$$f(x_1, x_2, \cdots, x_n) < f(y_1, y_2, \cdots, y_n) \tag{2.29}$$

whenever $x_i < y_i$ for $i = 1, 2, \cdots, n$. The following are strictly increasing functions,

$$f(x_1, x_2, \cdots, x_n) = x_1 \vee x_2 \vee \cdots \vee x_n,$$
$$f(x_1, x_2, \cdots, x_n) = x_1 \wedge x_2 \wedge \cdots \wedge x_n,$$
$$f(x_1, x_2, \cdots, x_n) = x_1 + x_2 + \cdots + x_n,$$
$$f(x_1, x_2, \cdots, x_n) = x_1 x_2 \cdots x_n, \quad x_1, x_2, \cdots, x_n \geq 0.$$

Theorem 2.9 *(Liu [129]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If f is a strictly increasing function, then the uncertain variable*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{2.30}$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)). \tag{2.31}$$

Proof: For simplicity, we only prove the case $n = 2$. At first, we always have

$$\{\xi \leq \Psi^{-1}(\alpha)\} \equiv \{f(\xi_1, \xi_2) \leq f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha))\}.$$

On the one hand, since f is a strictly increasing function, we obtain

$$\{\xi \leq \Psi^{-1}(\alpha)\} \supset \{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cap \{\xi_2 \leq \Phi_2^{-1}(\alpha)\}.$$

By using the independence of ξ_1 and ξ_2, we get

$$\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} \geq \mathcal{M}\{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \wedge \mathcal{M}\{\xi_2 \leq \Phi_2^{-1}(\alpha)\} = \alpha \wedge \alpha = \alpha.$$

On the other hand, since f is a strictly increasing function, we obtain

$$\{\xi \leq \Psi^{-1}(\alpha)\} \subset \{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cup \{\xi_2 \leq \Phi_2^{-1}(\alpha)\}.$$

By using the independence of ξ_1 and ξ_2, we get

$$\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} \leq \mathcal{M}\{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \vee \mathcal{M}\{\xi_2 \leq \Phi_2^{-1}(\alpha)\} = \alpha \vee \alpha = \alpha.$$

It follows that $\mathcal{M}\{\xi \leq \Psi^{-1}(\alpha)\} = \alpha$. That is, Ψ^{-1} is just the inverse uncertainty distribution of ξ. The theorem is proved.

Exercise 2.5: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Show that the sum

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n \tag{2.32}$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha). \tag{2.33}$$

Exercise 2.6: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent and positive uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Show that the product

$$\xi = \xi_1 \times \xi_2 \times \cdots \times \xi_n \tag{2.34}$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \times \Phi_2^{-1}(\alpha) \times \cdots \times \Phi_n^{-1}(\alpha). \tag{2.35}$$

Exercise 2.7: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Show that the minimum

$$\xi = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n \tag{2.36}$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \wedge \Phi_2^{-1}(\alpha) \wedge \cdots \wedge \Phi_n^{-1}(\alpha). \tag{2.37}$$

Exercise 2.8: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Show that the maximum

$$\xi = \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n \tag{2.38}$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) \vee \Phi_2^{-1}(\alpha) \vee \cdots \vee \Phi_n^{-1}(\alpha). \tag{2.39}$$

Theorem 2.10 *Assume that ξ_1 and ξ_2 are independent linear uncertain variables $\mathcal{L}(a_1, b_1)$ and $\mathcal{L}(a_2, b_2)$, respectively. Then the sum $\xi_1 + \xi_2$ is also a linear uncertain variable $\mathcal{L}(a_1 + a_2, b_1 + b_2)$, i.e.,*

$$\mathcal{L}(a_1, b_1) + \mathcal{L}(a_2, b_2) = \mathcal{L}(a_1 + a_2, b_1 + b_2). \tag{2.40}$$

The product of a linear uncertain variable $\mathcal{L}(a, b)$ and a scalar number $k > 0$ is also a linear uncertain variable $\mathcal{L}(ka, kb)$, i.e.,

$$k \cdot \mathcal{L}(a, b) = \mathcal{L}(ka, kb). \tag{2.41}$$

Proof: Assume that the uncertain variables ξ_1 and ξ_2 have uncertainty distributions Φ_1 and Φ_2, respectively. Then

$$\Phi_1^{-1}(\alpha) = (1 - \alpha)a_1 + \alpha b_1,$$

$$\Phi_2^{-1}(\alpha) = (1 - \alpha)a_2 + \alpha b_2.$$

It follows from the operational law that the inverse uncertainty distribution of $\xi_1 + \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) = (1 - \alpha)(a_1 + a_2) + \alpha(b_1 + b_2).$$

Hence the sum is also a linear uncertain variable $\mathcal{L}(a_1 + a_2, b_1 + b_2)$. The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable $\xi \sim \mathcal{L}(a, b)$ is Φ. It follows from the operational law that when $k > 0$, the inverse uncertainty distribution of $k\xi$ is

$$\Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha) = (1 - \alpha)(ka) + \alpha(kb).$$

Hence $k\xi$ is just a linear uncertain variable $\mathcal{L}(ka, kb)$.

Theorem 2.11 *Assume that ξ_1 and ξ_2 are independent zigzag uncertain variables $\mathcal{Z}(a_1, b_1, c_1)$ and $\mathcal{Z}(a_2, b_2, c_3)$, respectively. Then the sum $\xi_1 + \xi_2$ is also a zigzag uncertain variable $\mathcal{Z}(a_1 + a_2, b_1 + b_2, c_1 + c_2)$, i.e.,*

$$\mathcal{Z}(a_1, b_1, c_1) + \mathcal{Z}(a_2, b_2, c_2) = \mathcal{Z}(a_1 + a_2, b_1 + b_2, c_1 + c_2). \tag{2.42}$$

The product of a zigzag uncertain variable $\mathcal{Z}(a, b, c)$ and a scalar number $k > 0$ is also a zigzag uncertain variable $\mathcal{Z}(ka, kb, kc)$, i.e.,

$$k \cdot \mathcal{Z}(a, b, c) = \mathcal{Z}(ka, kb, kc). \tag{2.43}$$

Proof: Assume that the uncertain variables ξ_1 and ξ_2 have uncertainty distributions Φ_1 and Φ_2, respectively. Then

$$\Phi_1^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)a_1 + 2\alpha b_1, & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)b_1 + (2\alpha - 1)c_1, & \text{if } \alpha \ge 0.5, \end{cases}$$

$$\Phi_2^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)a_2 + 2\alpha b_2, & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)b_2 + (2\alpha - 1)c_2, & \text{if } \alpha \ge 0.5. \end{cases}$$

It follows from the operational law that the inverse uncertainty distribution of $\xi_1 + \xi_2$ is

$$\Psi^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)(a_1 + a_2) + 2\alpha(b_1 + b_2), & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)(b_1 + b_2) + (2\alpha - 1)(c_1 + c_2), & \text{if } \alpha \ge 0.5. \end{cases}$$

Hence the sum is also a zigzag uncertain variable $\mathcal{Z}(a_1 + a_2, b_1 + b_2, c_1 + c_2)$. The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable $\xi \sim \mathcal{Z}(a, b, c)$ is Φ. It follows from the operational law that when $k > 0$, the inverse uncertainty distribution of $k\xi$ is

$$\Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)(ka) + 2\alpha(kb), & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)(kb) + (2\alpha - 1)(kc), & \text{if } \alpha \ge 0.5. \end{cases}$$

Hence $k\xi$ is just a zigzag uncertain variable $\mathcal{Z}(ka, kb, kc)$.

Theorem 2.12 *Let ξ_1 and ξ_2 be independent normal uncertain variables $\mathcal{N}(e_1, \sigma_1)$ and $\mathcal{N}(e_2, \sigma_2)$, respectively. Then the sum $\xi_1 + \xi_2$ is also a normal uncertain variable $\mathcal{N}(e_1 + e_2, \sigma_1 + \sigma_2)$, i.e.,*

$$\mathcal{N}(e_1, \sigma_1) + \mathcal{N}(e_2, \sigma_2) = \mathcal{N}(e_1 + e_2, \sigma_1 + \sigma_2). \tag{2.44}$$

The product of a normal uncertain variable $\mathcal{N}(e, \sigma)$ and a scalar number $k > 0$ is also a normal uncertain variable $\mathcal{N}(ke, k\sigma)$, i.e.,

$$k \cdot \mathcal{N}(e, \sigma) = \mathcal{N}(ke, k\sigma). \tag{2.45}$$

Proof: Assume that the uncertain variables ξ_1 and ξ_2 have uncertainty distributions Φ_1 and Φ_2, respectively. Then

$$\Phi_1^{-1}(\alpha) = e_1 + \frac{\sigma_1\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha},$$

$$\Phi_2^{-1}(\alpha) = e_2 + \frac{\sigma_2\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

It follows from the operational law that the inverse uncertainty distribution of $\xi_1 + \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) = (e_1 + e_2) + \frac{(\sigma_1 + \sigma_2)\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

Hence the sum is also a normal uncertain variable $\mathcal{N}(e_1 + e_2, \sigma_1 + \sigma_2)$. The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ is Φ. It follows from the operational law that, when $k > 0$, the inverse uncertainty distribution of $k\xi$ is

$$\Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha) = (ke) + \frac{(k\sigma)\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}.$$

Hence $k\xi$ is just a normal uncertain variable $\mathcal{N}(ke, k\sigma)$.

Theorem 2.13 *Assume that ξ_1 and ξ_2 are independent lognormal uncertain variables $\mathcal{LOGN}(e_1, \sigma_1)$ and $\mathcal{LOGN}(e_2, \sigma_2)$, respectively. Then the product $\xi_1 \cdot \xi_2$ is also a lognormal uncertain variable $\mathcal{LOGN}(e_1 + e_2, \sigma_1 + \sigma_2)$, i.e.,*

$$\mathcal{LOGN}(e_1, \sigma_1) \cdot \mathcal{LOGN}(e_2, \sigma_2) = \mathcal{LOGN}(e_1 + e_2, \sigma_1 + \sigma_2). \tag{2.46}$$

The product of a lognormal uncertain variable $\mathcal{LOGN}(e, \sigma)$ and a scalar number $k > 0$ is also a lognormal uncertain variable $\mathcal{LOGN}(e + \ln k, \sigma)$, i.e.,

$$k \cdot \mathcal{LOGN}(e, \sigma) = \mathcal{LOGN}(e + \ln k, \sigma). \tag{2.47}$$

Proof: Assume that the uncertain variables ξ_1 and ξ_2 have uncertainty distributions Φ_1 and Φ_2, respectively. Then

$$\Phi_1^{-1}(\alpha) = \exp\left(e_1 + \frac{\sigma_1\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right),$$

$$\Phi_2^{-1}(\alpha) = \exp\left(e_2 + \frac{\sigma_2\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

It follows from the operational law that the inverse uncertainty distribution of $\xi_1 \cdot \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha)\cdot\Phi_2^{-1}(\alpha) = \exp\left((e_1+e_2) + \frac{(\sigma_1+\sigma_2)\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

Hence the product is a lognormal uncertain variable $\mathcal{LOGN}(e_1+e_2,\sigma_1+\sigma_2)$. The first part is verified. Next, suppose that the uncertainty distribution of the uncertain variable $\xi \sim \mathcal{LOGN}(e,\sigma)$ is Φ. It follows from the operational law that, when $k > 0$, the inverse uncertainty distribution of $k\xi$ is

$$\Psi^{-1}(\alpha) = k\Phi^{-1}(\alpha) = \exp\left((e+\ln k) + \frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right).$$

Hence $k\xi$ is just a lognormal uncertain variable $\mathcal{LOGN}(e+\ln k,\sigma)$.

Theorem 2.14 *(Liu [129]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If f is a strictly increasing function, then the uncertain variable*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{2.48}$$

has an uncertainty distribution

$$\Psi(x) = \sup_{f(x_1,x_2,\cdots,x_n)=x}\ \min_{1\le i\le n}\Phi_i(x_i). \tag{2.49}$$

Proof: For simplicity, we only prove the case $n = 2$. Since f is a strictly increasing function, it holds that

$$\{f(\xi_1,\xi_2)\le x\} = \bigcup_{f(x_1,x_2)=x}(\xi_1\le x_1)\cap(\xi_2\le x_2).$$

Thus the uncertainty distribution is

$$\Psi(x) = \mathcal{M}\{f(\xi_1,\xi_2)\le x\} = \mathcal{M}\left\{\bigcup_{f(x_1,x_2)=x}(\xi_1\le x_1)\cap(\xi_2\le x_2)\right\}.$$

Note that for each given number x, the event

$$\bigcup_{f(x_1,x_2)=x} (\xi_1 \le x_1) \cap (\xi_2 \le x_2)$$

is just a polyrectangle. It follows from the polyrectangular theorem that

$$\begin{aligned}\Psi(x) &= \sup_{f(x_1,x_2)=x} \mathcal{M}\{(\xi_1 \le x_1) \cap (\xi_2 \le x_2)\}\\ &= \sup_{f(x_1,x_2)=x} \mathcal{M}\{\xi_1 \le x_1\} \wedge \mathcal{M}\{\xi_2 \le x_2\}\\ &= \sup_{f(x_1,x_2)=x} \Phi_1(x_1) \wedge \Phi_2(x_2).\end{aligned}$$

The theorem is proved.

Exercise 2.9: Let ξ be an uncertain variable with uncertainty distribution Φ, and let f be a strictly increasing function. Show that $f(\xi)$ has an uncertainty distribution

$$\Psi(x) = \Phi(f^{-1}(x)), \quad \forall x \in \Re. \tag{2.50}$$

Exercise 2.10: Let $\xi_1, \xi_2, \cdots, \xi_n$ be iid uncertain variables with a common uncertainty distribution Φ. Show that the sum

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n \tag{2.51}$$

has an uncertainty distribution

$$\Psi(x) = \Phi\left(\frac{x}{n}\right). \tag{2.52}$$

Exercise 2.11: Let $\xi_1, \xi_2, \cdots, \xi_n$ be iid and positive uncertain variables with a common uncertainty distribution Φ. Show that the product

$$\xi = \xi_1 \xi_2 \cdots \xi_n \tag{2.53}$$

has an uncertainty distribution

$$\Psi(x) = \Phi\left(\sqrt[n]{x}\right). \tag{2.54}$$

Exercise 2.12: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Show that the minimum

$$\xi = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n \tag{2.55}$$

has an uncertainty distribution

$$\Psi(x) = \Phi_1(x) \vee \Phi_2(x) \vee \cdots \vee \Phi_n(x). \tag{2.56}$$

Exercise 2.13: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Show that the maximum

$$\xi = \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n \tag{2.57}$$

has an uncertainty distribution

$$\Psi(x) = \Phi_1(x) \wedge \Phi_2(x) \wedge \cdots \wedge \Phi_n(x). \tag{2.58}$$

Theorem 2.15 *(Liu [135], Extreme Value Theorem) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables. Assume that*

$$S_i = \xi_1 + \xi_2 + \cdots + \xi_i \tag{2.59}$$

have uncertainty distributions Ψ_i for $i = 1, 2, \cdots, n$, respectively. Then the maximum

$$S = S_1 \vee S_2 \vee \cdots \vee S_n \tag{2.60}$$

has an uncertainty distribution

$$\Upsilon(x) = \Psi_1(x) \wedge \Psi_2(x) \wedge \cdots \wedge \Psi_n(x); \tag{2.61}$$

and the minimum

$$S = S_1 \wedge S_2 \wedge \cdots \wedge S_n \tag{2.62}$$

has an uncertainty distribution

$$\Upsilon(x) = \Psi_1(x) \vee \Psi_2(x) \vee \cdots \vee \Psi_n(x). \tag{2.63}$$

Proof: Assume that the uncertainty distributions of the uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ are $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Define

$$f(x_1, x_2, \cdots, x_n) = x_1 \vee (x_1 + x_2) \vee \cdots \vee (x_1 + x_2 + \cdots + x_n).$$

Then f is a strictly increasing function and

$$S = f(\xi_1, \xi_2, \cdots, \xi_n).$$

It follows from Theorem 2.14 that S has an uncertainty distribution

$$\begin{aligned}
\Upsilon(x) &= \sup_{f(x_1, x_2, \cdots, x_n) = x} \Phi_1(x_1) \wedge \Phi_2(x_2) \wedge \cdots \wedge \Phi_n(x_n) \\
&= \min_{1 \le i \le n} \sup_{x_1 + x_2 + \cdots + x_i = x} \Phi_1(x_1) \wedge \Phi_2(x_2) \wedge \cdots \wedge \Phi_i(x_i) \\
&= \min_{1 \le i \le n} \Psi_i(x).
\end{aligned}$$

Thus (2.61) is verified. Similarly, define

$$f(x_1, x_2, \cdots, x_n) = x_1 \wedge (x_1 + x_2) \wedge \cdots \wedge (x_1 + x_2 + \cdots + x_n).$$

Then f is a strictly increasing function and

$$S = f(\xi_1, \xi_2, \cdots, \xi_n).$$

It follows from Theorem 2.14 that S has an uncertainty distribution

$$\begin{aligned}\Upsilon(x) &= \sup_{f(x_1,x_2,\cdots,x_n)=x} \Phi_1(x_1) \wedge \Phi_2(x_2) \wedge \cdots \wedge \Phi_n(x_n) \\ &= \max_{1\le i\le n} \sup_{x_1+x_2+\cdots+x_i=x} \Phi_1(x_1) \wedge \Phi_2(x_2) \wedge \cdots \wedge \Phi_i(x_i) \\ &= \max_{1\le i\le n} \Psi_i(x).\end{aligned}$$

Thus (2.63) is verified.

Strictly Decreasing Function of Uncertain Variables

A real-valued function $f(x_1, x_2, \cdots, x_n)$ is said to be strictly decreasing if

$$f(x_1, x_2, \cdots, x_n) \ge f(y_1, y_2, \cdots, y_n) \tag{2.64}$$

whenever $x_i \le y_i$ for $i = 1, 2, \cdots, n$, and

$$f(x_1, x_2, \cdots, x_n) > f(y_1, y_2, \cdots, y_n) \tag{2.65}$$

whenever $x_i < y_i$ for $i = 1, 2, \cdots, n$. If $f(x_1, x_2, \cdots, x_n)$ is a strictly increasing function, then $-f(x_1, x_2, \cdots, x_n)$ is a strictly decreasing function. Furthermore, $1/f(x_1, x_2, \cdots, x_n)$ is also a strictly decreasing function provided that f is positive. Especially, the following are strictly decreasing functions,

$$f(x) = -x,$$

$$f(x) = \exp(-x),$$

$$f(x) = \frac{1}{x}, \quad x > 0.$$

Theorem 2.16 *(Liu [129]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If f is a strictly decreasing function, then the uncertain variable*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{2.66}$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)). \tag{2.67}$$

Proof: For simplicity, we only prove the case $n = 2$. At first, we always have

$$\{\xi \le \Psi^{-1}(\alpha)\} \equiv \{f(\xi_1, \xi_2) \le f(\Phi_1^{-1}(1-\alpha), \Phi_2^{-1}(1-\alpha))\}.$$

On the one hand, since f is a strictly decreasing function, we obtain

$$\{\xi \le \Psi^{-1}(\alpha)\} \supset \{\xi_1 \ge \Phi_1^{-1}(1-\alpha)\} \cap \{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\}.$$

By using the independence of ξ_1 and ξ_2, we get

$$\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} \ge \mathcal{M}\{\xi_1 \ge \Phi_1^{-1}(1-\alpha)\} \wedge \mathcal{M}\{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\} = \alpha \wedge \alpha = \alpha.$$

On the other hand, since f is a strictly decreasing function, we obtain

$$\{\xi \le \Psi^{-1}(\alpha)\} \subset \{\xi_1 \ge \Phi_1^{-1}(1-\alpha)\} \cup \{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\}.$$

By using the independence of ξ_1 and ξ_2, we get

$$\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} \le \mathcal{M}\{\xi_1 \ge \Phi_1^{-1}(1-\alpha)\} \vee \mathcal{M}\{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\} = \alpha \vee \alpha = \alpha.$$

It follows that $\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} = \alpha$. That is, Ψ^{-1} is just the inverse uncertainty distribution of ξ. The theorem is proved.

Exercise 2.14: Let ξ be a positive uncertain variable with regular uncertainty distribution Φ. Show that the reciprocal $1/\xi$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \frac{1}{\Phi^{-1}(1-\alpha)}. \tag{2.68}$$

Exercise 2.15: Let ξ be an uncertain variable with regular uncertainty distribution Φ. Show that $\exp(-\xi)$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = \exp\left(-\Phi^{-1}(1-\alpha)\right). \tag{2.69}$$

Theorem 2.17 *(Liu [129]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with continuous uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If f is a strictly decreasing function, then the uncertain variable*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{2.70}$$

has an uncertainty distribution

$$\Psi(x) = \sup_{f(x_1,x_2,\cdots,x_n)=x} \min_{1\le i\le n} (1 - \Phi_i(x_i)). \tag{2.71}$$

Proof: For simplicity, we only prove the case $n = 2$. Since f is a strictly decreasing function, it holds that

$$\{f(\xi_1, \xi_2) \le x\} = \bigcup_{f(x_1,x_2)=x} (\xi_1 \ge x_1) \cap (\xi_2 \ge x_2).$$

Thus the uncertainty distribution is

$$\Psi(x) = \mathcal{M}\{f(\xi_1, \xi_2) \le x\} = \mathcal{M}\left\{\bigcup_{f(x_1,x_2)=x} (\xi_1 \ge x_1) \cap (\xi_2 \ge x_2)\right\}.$$

Note that for each given number x, the event

$$\bigcup_{f(x_1,x_2)=x} (\xi_1 \ge x_1) \cap (\xi_2 \ge x_2)$$

is just a polyrectangle. It follows from the polyrectangular theorem that

$$\begin{aligned}\Psi(x) &= \sup_{f(x_1,x_2)=x} \mathcal{M}\left\{(\xi_1 \ge x_1) \cap (\xi_2 \ge x_2)\right\}\\ &= \sup_{f(x_1,x_2)=x} \mathcal{M}\{\xi_1 \ge x_1\} \wedge \mathcal{M}\{\xi_2 \ge x_2\}\\ &= \sup_{f(x_1,x_2)=x} (1-\Phi_1(x_1)) \wedge (1-\Phi_2(x_2)).\end{aligned}$$

The theorem is proved.

Exercise 2.16: Let ξ be an uncertain variable with continuous uncertainty distribution Φ, and let f be a strictly decreasing function. Show that $f(\xi)$ has an uncertainty distribution

$$\Psi(x) = 1 - \Phi(f^{-1}(x)), \quad \forall x \in \Re. \tag{2.72}$$

Exercise 2.17: Let ξ be an uncertain variable with continuous uncertainty distribution Φ, and let a and b be real numbers with $a < 0$. Show that $a\xi + b$ has an uncertainty distribution

$$\Psi(x) = 1 - \Phi\left(\frac{x-b}{a}\right), \quad \forall x \in \Re. \tag{2.73}$$

Exercise 2.18: Let ξ be a positive uncertain variable with continuous uncertainty distribution Φ. Show that $1/\xi$ has an uncertainty distribution

$$\Psi(x) = 1 - \Phi\left(\frac{1}{x}\right), \quad \forall x > 0. \tag{2.74}$$

Exercise 2.19: Let ξ be an uncertain variable with continuous uncertainty distribution Φ. Show that $\exp(-\xi)$ has an uncertainty distribution

$$\Psi(x) = 1 - \Phi(-\ln(x)), \quad \forall x > 0. \tag{2.75}$$

Strictly Monotone Function of Uncertain Variables

A real-valued function $f(x_1, x_2, \cdots, x_n)$ is said to be strictly monotone if it is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, that is,

$$f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) \le f(y_1, \cdots, y_m, y_{m+1}, \cdots, y_n) \tag{2.76}$$

whenever $x_i \le y_i$ for $i = 1, 2, \cdots, m$ and $x_i \ge y_i$ for $i = m+1, m+2, \cdots, n$, and

$$f(x_1, \cdots, x_m, x_{m+1}, \cdots, x_n) < f(y_1, \cdots, y_m, y_{m+1}, \cdots, y_n) \tag{2.77}$$

whenever $x_i < y_i$ for $i = 1, 2, \cdots, m$ and $x_i > y_i$ for $i = m+1, m+2, \cdots, n$. The following are strictly monotone functions,

$$\begin{aligned} f(x_1, x_2) &= x_1 - x_2, \\ f(x_1, x_2) &= x_1/x_2, \quad x_1, x_2 > 0, \\ f(x_1, x_2) &= x_1/(x_1 + x_2), \quad x_1, x_2 > 0. \end{aligned}$$

Note that both strictly increasing function and strictly decreasing function are special cases of strictly monotone function.

Theorem 2.18 *(Liu [129]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then the uncertain variable*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{2.78}$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)). \tag{2.79}$$

Proof: We only prove the case of $m = 1$ and $n = 2$. At first, we always have

$$\{\xi \le \Psi^{-1}(\alpha)\} \equiv \{f(\xi_1, \xi_2) \le f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1-\alpha))\}.$$

On the one hand, since the function $f(x_1, x_2)$ is strictly increasing with respect to x_1 and strictly decreasing with x_2, we obtain

$$\{\xi \le \Psi^{-1}(\alpha)\} \supset \{\xi_1 \le \Phi_1^{-1}(\alpha)\} \cap \{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\}.$$

By using the independence of ξ_1 and ξ_2, we get

$$\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} \ge \mathcal{M}\{\xi_1 \le \Phi_1^{-1}(\alpha)\} \wedge \mathcal{M}\{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\} = \alpha \wedge \alpha = \alpha.$$

On the other hand, since the function $f(x_1, x_2)$ is strictly increasing with respect to x_1 and strictly decreasing with x_2, we obtain

$$\{\xi \le \Psi^{-1}(\alpha)\} \subset \{\xi_1 \le \Phi_1^{-1}(\alpha)\} \cup \{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\}.$$

By using the independence of ξ_1 and ξ_2, we get

$$\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} \le \mathcal{M}\{\xi_1 \le \Phi_1^{-1}(\alpha)\} \vee \mathcal{M}\{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\} = \alpha \vee \alpha = \alpha.$$

It follows that $\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} = \alpha$. That is, Ψ^{-1} is just the inverse uncertainty distribution of ξ. The theorem is proved.

Exercise 2.20: Let ξ_1 and ξ_2 be independent uncertain variables with regular uncertainty distributions Φ_1 and Φ_2, respectively. Show that the inverse uncertainty distribution of the difference $\xi_1 - \xi_2$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) - \Phi_2^{-1}(1-\alpha). \tag{2.80}$$

Exercise 2.21: Let ξ_1 and ξ_2 be independent and positive uncertain variables with regular uncertainty distributions Φ_1 and Φ_2, respectively. Show that the inverse uncertainty distribution of the quotient ξ_1/ξ_2 is

$$\Psi^{-1}(\alpha) = \frac{\Phi_1^{-1}(\alpha)}{\Phi_2^{-1}(1-\alpha)}. \tag{2.81}$$

Exercise 2.22: Assume ξ_1 and ξ_2 are independent and positive uncertain variables with regular uncertainty distributions Φ_1 and Φ_2, respectively. Show that the inverse uncertainty distribution of $\xi_1/(\xi_1+\xi_2)$ is

$$\Psi^{-1}(\alpha) = \frac{\Phi_1^{-1}(\alpha)}{\Phi_1^{-1}(\alpha) + \Phi_2^{-1}(1-\alpha)}. \tag{2.82}$$

Theorem 2.19 *(Liu [129]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with continuous uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then the uncertain variable*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{2.83}$$

has an uncertainty distribution

$$\Psi(x) = \sup_{f(x_1,x_2,\cdots,x_n)=x} \left(\min_{1\le i\le m} \Phi_i(x_i) \wedge \min_{m+1\le i\le n} (1-\Phi_i(x_i)) \right). \tag{2.84}$$

Proof: For simplicity, we only prove the case of $m = 1$ and $n = 2$. Since $f(x_1, x_2)$ is strictly increasing with respect to x_1 and strictly decreasing with respect to x_2, it holds that

$$\{f(\xi_1,\xi_2) \le x\} = \bigcup_{f(x_1,x_2)=x} (\xi_1 \le x_1) \cap (\xi_2 \ge x_2).$$

Thus the uncertainty distribution is

$$\Psi(x) = \mathcal{M}\{f(\xi_1,\xi_2) \le x\} = \mathcal{M}\left\{ \bigcup_{f(x_1,x_2)=x} (\xi_1 \le x_1) \cap (\xi_2 \ge x_2) \right\}.$$

Note that for each given number x, the event

$$\bigcup_{f(x_1,x_2)=x} (\xi_1 \le x_1) \cap (\xi_2 \ge x_2)$$

is just a polyrectangle. It follows from the polyrectangular theorem that

$$\begin{aligned}\Psi(x) &= \sup_{f(x_1,x_2)=x} \mathcal{M}\left\{(\xi_1 \le x_1) \cap (\xi_2 \ge x_2)\right\}\\ &= \sup_{f(x_1,x_2)=x} \mathcal{M}\{\xi_1 \le x_1\} \wedge \mathcal{M}\{\xi_2 \ge x_2\}\\ &= \sup_{f(x_1,x_2)=x} \Phi_1(x_1) \wedge (1 - \Phi_2(x_2)).\end{aligned}$$

The theorem is proved.

Exercise 2.23: Let ξ_1 and ξ_2 be independent uncertain variables with continuous uncertainty distributions Φ_1 and Φ_2, respectively. Show that $\xi_1 - \xi_2$ has an uncertainty distribution

$$\Psi(x) = \sup_{y \in \Re} \Phi_1(x+y) \wedge (1 - \Phi_2(y)). \tag{2.85}$$

Exercise 2.24: Let ξ_1 and ξ_2 be independent and positive uncertain variables with continuous uncertainty distributions Φ_1 and Φ_2, respectively. Show that ξ_1/ξ_2 has an uncertainty distribution

$$\Psi(x) = \sup_{y>0} \Phi_1(xy) \wedge (1 - \Phi_2(y)). \tag{2.86}$$

Exercise 2.25: Let ξ_1 and ξ_2 be independent and positive uncertain variables with continuous uncertainty distributions Φ_1 and Φ_2, respectively. Show that $\xi_1/(\xi_1 + \xi_2)$ has an uncertainty distribution

$$\Psi(x) = \sup_{y>0} \Phi_1(xy) \wedge (1 - \Phi_2(y - xy)). \tag{2.87}$$

Some Useful Theorems

In many cases, it is required to calculate $\mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) \le 0\}$. Perhaps the first idea is to produce the uncertainty distribution $\Psi(x)$ of $f(\xi_1, \xi_2, \cdots, \xi_n)$ by the operational law, and then the uncertain measure is just $\Psi(0)$. However, for convenience, we may use the following theorems.

Theorem 2.20 *(Liu [128]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If*

$f(\xi_1,\xi_2,\cdots,\xi_n)$ is strictly increasing with respect to $\xi_1,\xi_2,\cdots,\xi_m$ and strictly decreasing with respect to $\xi_{m+1},\xi_{m+2},\cdots,\xi_n$, then

$$\mathcal{M}\{f(\xi_1,\xi_2,\cdots,\xi_n)\le 0\} \tag{2.88}$$

is just the root α of the equation

$$f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha))=0. \tag{2.89}$$

Proof: It follows from Theorem 2.18 that $f(\xi_1,\xi_2,\cdots,\xi_n)$ is an uncertain variable whose inverse uncertainty distribution is

$$\Psi^{-1}(\alpha)=f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha)).$$

Since $\mathcal{M}\{f(\xi_1,\xi_2,\cdots,\xi_n)\le 0\}=\Psi(0)$, it is the solution α of the equation $\Psi^{-1}(\alpha)=0$. The theorem is proved.

Remark 2.3: Keep in mind that sometimes the equation (2.89) may not have a root. In this case, if

$$f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha))<0 \tag{2.90}$$

for all α, then we set the root $\alpha=1$; and if

$$f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha))>0 \tag{2.91}$$

for all α, then we set the root $\alpha=0$.

Remark 2.4: Since $f(\xi_1,\xi_2,\cdots,\xi_n)$ is strictly increasing with respect to $\xi_1,\xi_2,\cdots,\xi_m$ and strictly decreasing with respect to $\xi_{m+1},\xi_{m+2},\cdots,\xi_n$, the function $f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha))$ is a strictly increasing function with respect to α. See Figure 2.14. Thus its root α may be estimated by the bisection method:

Step 1. Set $a=0$, $b=1$ and $c=(a+b)/2$.

Step 2. If $f(\Phi_1^{-1}(c),\cdots,\Phi_m^{-1}(c),\Phi_{m+1}^{-1}(1-c),\cdots,\Phi_n^{-1}(1-c))\le 0$, then set $a=c$. Otherwise, set $b=c$.

Step 3. If $|b-a|>\varepsilon$ (a predetermined precision), then set $c=(b-a)/2$ and go to Step 2. Otherwise, output b as the root.

Theorem 2.21 *(Liu [128]) Let $\xi_1,\xi_2,\cdots,\xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\cdots,\Phi_n$, respectively. If $f(\xi_1,\xi_2,\cdots,\xi_n)$ is strictly increasing with respect to $\xi_1,\xi_2,\cdots,\xi_m$ and strictly decreasing with respect to $\xi_{m+1},\xi_{m+2},\cdots,\xi_n$, then*

$$\mathcal{M}\{f(\xi_1,\xi_2,\cdots,\xi_n)>0\} \tag{2.92}$$

is just the root α of the equation

$$f(\Phi_1^{-1}(1-\alpha),\cdots,\Phi_m^{-1}(1-\alpha),\Phi_{m+1}^{-1}(\alpha),\cdots,\Phi_n^{-1}(\alpha))=0. \tag{2.93}$$

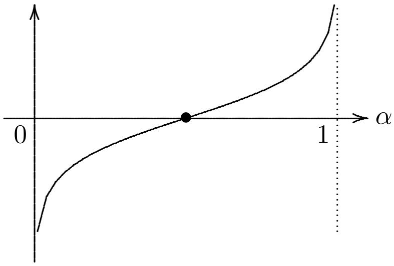

Figure 2.14: $f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha))$

Proof: It follows from Theorem 2.18 that $f(\xi_1,\xi_2,\cdots,\xi_n)$ is an uncertain variable whose inverse uncertainty distribution is

$$\Psi^{-1}(\alpha)=f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha)).$$

Since $\mathcal{M}\{f(\xi_1,\xi_2,\cdots,\xi_n)>0\}=1-\Psi(0)$, it is the solution α of the equation $\Psi^{-1}(1-\alpha)=0$. The theorem is proved.

Remark 2.5: Keep in mind that sometimes the equation (2.93) may not have a root. In this case, if

$$f(\Phi_1^{-1}(1-\alpha),\cdots,\Phi_m^{-1}(1-\alpha),\Phi_{m+1}^{-1}(\alpha),\cdots,\Phi_n^{-1}(\alpha))<0 \tag{2.94}$$

for all α, then we set the root $\alpha=0$; and if

$$f(\Phi_1^{-1}(1-\alpha),\cdots,\Phi_m^{-1}(1-\alpha),\Phi_{m+1}^{-1}(\alpha),\cdots,\Phi_n^{-1}(\alpha))>0 \tag{2.95}$$

for all α, then we set the root $\alpha=1$.

Remark 2.6: Since $f(\xi_1,\xi_2,\cdots,\xi_n)$ is strictly increasing with respect to $\xi_1,\xi_2,\cdots,\xi_m$ and strictly decreasing with respect to $\xi_{m+1},\xi_{m+2},\cdots,\xi_n$, the function $f(\Phi_1^{-1}(1-\alpha),\cdots,\Phi_m^{-1}(1-\alpha),\Phi_{m+1}^{-1}(\alpha),\cdots,\Phi_n^{-1}(\alpha))$ is a strictly decreasing function with respect to α. See Figure 2.15. Thus its root α may be estimated by the bisection method:

Step 1. Set $a=0$, $b=1$ and $c=(a+b)/2$.

Step 2. If $f(\Phi_1^{-1}(1-c),\cdots,\Phi_m^{-1}(1-c),\Phi_{m+1}^{-1}(c),\cdots,\Phi_n^{-1}(c))>0$, then set $a=c$. Otherwise, set $b=c$.

Step 3. If $|b-a|>\varepsilon$ (a predetermined precision), then set $c=(b-a)/2$ and go to Step 2. Otherwise, output b as the root.

Theorem 2.22 *Let $\xi_1,\xi_2,\cdots,\xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\cdots,\Phi_n$, respectively. If the function $f(\xi_1,\xi_2,\cdots,\xi_n)$ is strictly increasing with respect to $\xi_1,\xi_2,\cdots,\xi_m$ and*

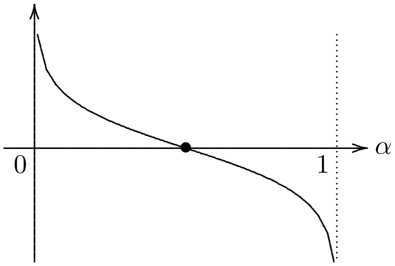

Figure 2.15: $f(\Phi_1^{-1}(1-\alpha),\cdots,\Phi_m^{-1}(1-\alpha),\Phi_{m+1}^{-1}(\alpha),\cdots,\Phi_n^{-1}(\alpha))$

strictly decreasing with respect to $\xi_{m+1},\xi_{m+2},\cdots,\xi_n$, *then*

$$\mathcal{M}\{f(\xi_1,\xi_2,\cdots,\xi_n)\le 0\}\ge\alpha \tag{2.96}$$

if and only if

$$f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha))\le 0. \tag{2.97}$$

Proof: It follows from Theorem 2.18 that the inverse uncertainty distribution of $f(\xi_1,\xi_2,\cdots,\xi_n)$ is

$$\Psi^{-1}(\alpha)=f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha)).$$

Thus (2.96) holds if and only if $\Psi^{-1}(\alpha)\le 0$. The theorem is thus verified.

Boolean Function of Boolean Uncertain Variables

A function is said to be Boolean if it is a mapping from $\{0,1\}^n$ to $\{0,1\}$. For example,

$$f(x_1,x_2,x_3)=x_1\vee x_2\wedge x_3 \tag{2.98}$$

is a Boolean function. An uncertain variable is said to be Boolean if it takes values either 0 or 1. For example, the following is a Boolean uncertain variable,

$$\xi=\begin{cases}1 \text{ with uncertain measure } a\\ 0 \text{ with uncertain measure } 1-a\end{cases} \tag{2.99}$$

where a is a number between 0 and 1. This subsection introduces an operational law for Boolean system.

Theorem 2.23 *(Liu [129]) Assume* $\xi_1,\xi_2,\cdots,\xi_n$ *are independent Boolean uncertain variables, i.e.,*

$$\xi_i=\begin{cases}\textit{1 with uncertain measure } a_i\\ \textit{0 with uncertain measure } 1-a_i\end{cases} \tag{2.100}$$

for $i = 1, 2, \cdots, n$. If f is a Boolean function (not necessarily monotone), then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = \begin{cases} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \text{if} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \sup\limits_{f(x_1,x_2,\cdots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \text{if} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) \ge 0.5 \end{cases} \tag{2.101}$$

where x_i take values either 0 or 1, and ν_i are defined by

$$\nu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \tag{2.102}$$

for $i = 1, 2, \cdots, n$, respectively.

Proof: Let $B_1, B_2, \cdots, B_n$ be nonempty subsets of $\{0, 1\}$. In other words, they take values of $\{0\}$, $\{1\}$ or $\{0, 1\}$. Write

$$\Lambda = \{\xi = 1\}, \quad \Lambda^c = \{\xi = 0\}, \quad \Lambda_i = \{\xi_i \in B_i\}$$

for $i = 1, 2, \cdots, n$. It is easy to verify that

$$\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n = \Lambda \text{ if and only if } f(B_1, B_2, \cdots, B_n) = \{1\},$$

$$\Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_n = \Lambda^c \text{ if and only if } f(B_1, B_2, \cdots, B_n) = \{0\}.$$

It follows from the product axiom that

$$\mathcal{M}\{\xi = 1\} = \begin{cases} \sup\limits_{f(B_1,B_2,\cdots,B_n)=\{1\}} \min\limits_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\}, \\ \qquad \text{if} \sup\limits_{f(B_1,B_2,\cdots,B_n)=\{1\}} \min\limits_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} > 0.5 \\ 1 - \sup\limits_{f(B_1,B_2,\cdots,B_n)=\{0\}} \min\limits_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\}, \\ \qquad \text{if} \sup\limits_{f(B_1,B_2,\cdots,B_n)=\{0\}} \min\limits_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} > 0.5 \\ 0.5, \ \text{otherwise.} \end{cases} \tag{2.103}$$

Please note that

$$\nu_i(1) = \mathcal{M}\{\xi_i = 1\}, \quad \nu_i(0) = \mathcal{M}\{\xi_i = 0\}$$

for $i = 1, 2, \cdots, n$. The argument breaks down into four cases. Case 1: Assume

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) < 0.5.$$

Then we have

$$\sup_{f(B_1,B_2,\cdots,B_n)=\{0\}} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) > 0.5.$$

It follows from (2.103) that

$$\mathcal{M}\{\xi = 1\} = \sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i).$$

Case 2: Assume

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) > 0.5.$$

Then we have

$$\sup_{f(B_1,B_2,\cdots,B_n)=\{1\}} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) > 0.5.$$

It follows from (2.103) that

$$\mathcal{M}\{\xi = 1\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i).$$

Case 3: Assume

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) = 0.5,$$

$$\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) = 0.5.$$

Then we have

$$\sup_{f(B_1,B_2,\cdots,B_n)=\{1\}} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} = 0.5,$$

$$\sup_{f(B_1,B_2,\cdots,B_n)=\{0\}} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} = 0.5.$$

It follows from (2.103) that

$$\mathcal{M}\{\xi = 1\} = 0.5 = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i).$$

Case 4: Assume

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) = 0.5,$$

$$\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) < 0.5.$$

Then we have

$$\sup_{f(B_1,B_2,\cdots,B_n)=\{1\}} \min_{1\le i\le n} \mathcal{M}\{\xi_i \in B_i\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) > 0.5.$$

It follows from (2.103) that

$$\mathcal{M}\{\xi = 1\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i).$$

Hence the equation (2.101) is proved for the four cases.

Theorem 2.24 *Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables, i.e.,*

$$\xi_i = \begin{cases} \text{1 with uncertain measure } a_i \\ \text{0 with uncertain measure } 1 - a_i \end{cases} \tag{2.104}$$

for $i = 1, 2, \cdots, n$. Then the minimum

$$\xi = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n \tag{2.105}$$

is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = a_1 \wedge a_2 \wedge \cdots \wedge a_n, \tag{2.106}$$

$$\mathcal{M}\{\xi = 0\} = (1 - a_1) \vee (1 - a_2) \vee \cdots \vee (1 - a_n). \tag{2.107}$$

Proof: Since ξ is the minimum of Boolean uncertain variables, the corresponding Boolean function is

$$f(x_1, x_2, \cdots, x_n) = x_1 \wedge x_2 \wedge \cdots \wedge x_n. \tag{2.108}$$

Without loss of generality, we assume $a_1 \ge a_2 \ge \cdots \ge a_n$. Then we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1 \le i \le n} \nu_i(x_i) = \min_{1 \le i \le n} \nu_i(1) = a_n,$$

$$\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1 \le i \le n} \nu_i(x_i) = (1 - a_n) \wedge \min_{1 \le i < n} (a_i \vee (1 - a_i))$$

where $\nu_i(x_i)$ are defined by (2.102) for $i = 1, 2, \cdots, n$, respectively. When $a_n < 0.5$, we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1 \le i \le n} \nu_i(x_i) = a_n < 0.5.$$

It follows from Theorem 2.23 that

$$\mathcal{M}\{\xi = 1\} = \sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1 \le i \le n} \nu_i(x_i) = a_n.$$

When $a_n \ge 0.5$, we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1 \le i \le n} \nu_i(x_i) = a_n \ge 0.5.$$

It follows from Theorem 2.23 that

$$\mathcal{M}\{\xi = 1\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1 \le i \le n} \nu_i(x_i) = 1 - (1 - a_n) = a_n.$$

Thus $\mathcal{M}\{\xi = 1\}$ is always a_n, i.e., the minimum value of $a_1, a_2, \cdots, a_n$. Thus the equation (2.106) is proved. The equation (2.107) may be verified by the duality of uncertain measure.

Theorem 2.25 *Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables, i.e.,*

$$\xi_i = \begin{cases} \text{1 with uncertain measure } a_i \\ \text{0 with uncertain measure } 1 - a_i \end{cases} \tag{2.109}$$

for $i = 1, 2, \cdots, n$. Then the maximum

$$\xi = \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n \tag{2.110}$$

is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = a_1 \vee a_2 \vee \cdots \vee a_n, \tag{2.111}$$

$$\mathcal{M}\{\xi = 0\} = (1 - a_1) \wedge (1 - a_2) \wedge \cdots \wedge (1 - a_n). \tag{2.112}$$

Proof: Since ξ is the maximum of Boolean uncertain variables, the corresponding Boolean function is

$$f(x_1, x_2, \cdots, x_n) = x_1 \vee x_2 \vee \cdots \vee x_n. \tag{2.113}$$

Without loss of generality, we assume $a_1 \ge a_2 \ge \cdots \ge a_n$. Then we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) = a_1 \wedge \min_{1<i\le n} (a_i \vee (1 - a_i)),$$

$$\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) = \min_{1\le i\le n} \nu_i(0) = 1 - a_1$$

where $\nu_i(x_i)$ are defined by (2.102) for $i = 1, 2, \cdots, n$, respectively. When $a_1 \ge 0.5$, we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) \ge 0.5.$$

It follows from Theorem 2.23 that

$$\mathcal{M}\{\xi = 1\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\le i\le n} \nu_i(x_i) = 1 - (1 - a_1) = a_1.$$

When $a_1 < 0.5$, we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) = a_1 < 0.5.$$

It follows from Theorem 2.23 that

$$\mathcal{M}\{\xi = 1\} = \sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) = a_1.$$

Thus $\mathcal{M}\{\xi = 1\}$ is always a_1, i.e., the maximum value of $a_1, a_2, \cdots, a_n$. Thus the equation (2.111) is proved. The equation (2.112) may be verified by the duality of uncertain measure.

Theorem 2.26 *Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables, i.e.,*

$$\xi_i = \begin{cases} \text{1 with uncertain measure } a_i \\ \text{0 with uncertain measure } 1 - a_i \end{cases} \tag{2.114}$$

for $i = 1, 2, \cdots, n$. Then

$$\xi = \begin{cases} 1, & \text{if } \xi_1 + \xi_2 + \cdots + \xi_n \geq k \\ 0, & \text{if } \xi_1 + \xi_2 + \cdots + \xi_n < k \end{cases} \tag{2.115}$$

is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = k\text{-max}\,[a_1, a_2, \cdots, a_n] \tag{2.116}$$

and

$$\mathcal{M}\{\xi = 0\} = k\text{-min}\,[1 - a_1, 1 - a_2, \cdots, 1 - a_n] \tag{2.117}$$

where k-max *represents the* k*th largest value, and* k-min *represents the* k*th smallest value.*

Proof: This is the so-called k-out-of-n system. The corresponding Boolean function is

$$f(x_1, x_2, \cdots, x_n) = \begin{cases} 1, & \text{if } x_1 + x_2 + \cdots + x_n \geq k \\ 0, & \text{if } x_1 + x_2 + \cdots + x_n < k. \end{cases} \tag{2.118}$$

Without loss of generality, we assume $a_1 \geq a_2 \geq \cdots \geq a_n$. Then we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\leq i\leq n} \nu_i(x_i) = a_k \wedge \min_{k<i\leq n} (a_i \vee (1 - a_i)),$$

$$\sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\leq i\leq n} \nu_i(x_i) = (1 - a_k) \wedge \min_{k<i\leq n} (a_i \vee (1 - a_i))$$

where $\nu_i(x_i)$ are defined by (2.102) for $i = 1, 2, \cdots, n$, respectively. When $a_k \geq 0.5$, we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\leq i\leq n} \nu_i(x_i) \geq 0.5.$$

It follows from Theorem 2.23 that

$$\mathcal{M}\{\xi = 1\} = 1 - \sup_{f(x_1,x_2,\cdots,x_n)=0} \min_{1\leq i\leq n} \nu_i(x_i) = 1 - (1 - a_k) = a_k.$$

When $a_k < 0.5$, we have

$$\sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\leq i\leq n} \nu_i(x_i) = a_k < 0.5.$$

It follows from Theorem 2.23 that

$$\mathcal{M}\{\xi = 1\} = \sup_{f(x_1,x_2,\cdots,x_n)=1} \min_{1\le i\le n} \nu_i(x_i) = a_k.$$

Thus $\mathcal{M}\{\xi = 1\}$ is always a_k, i.e., the kth largest value of $a_1, a_2, \cdots, a_n$. Thus the equation (2.116) is proved. The equation (2.117) may be verified by the duality of uncertain measure.

Boolean System Calculator

Boolean System Calculator is a function in the Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) for computing the uncertain measure like

$$\mathcal{M}\{f(\xi_1,\xi_2,\cdots,\xi_n) = 1\}, \quad \mathcal{M}\{f(\xi_1,\xi_2,\cdots,\xi_n) = 0\} \tag{2.119}$$

where $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables and f is a Boolean function. For example, let ξ_1, ξ_2, ξ_3 be independent Boolean uncertain variables,

$$\xi_1 = \begin{cases} 1 \text{ with uncertain mesure } 0.8 \\ 0 \text{ with uncertain mesure } 0.2, \end{cases}$$

$$\xi_2 = \begin{cases} 1 \text{ with uncertain mesure } 0.7 \\ 0 \text{ with uncertain mesure } 0.3, \end{cases}$$

$$\xi_3 = \begin{cases} 1 \text{ with uncertain mesure } 0.6 \\ 0 \text{ with uncertain mesure } 0.4. \end{cases}$$

We also assume the Boolean function is

$$f(x_1,x_2,x_3) = \begin{cases} 1, & \text{if } x_1 + x_2 + x_3 = 0 \text{ or } 2 \\ 0, & \text{if } x_1 + x_2 + x_3 = 1 \text{ or } 3. \end{cases}$$

The Boolean System Calculator yields $\mathcal{M}\{f(\xi_1,\xi_2,\xi_3) = 1\} = 0.4$.

2.5 Expected Value

Expected value is the average value of uncertain variable in the sense of uncertain measure, and represents the size of uncertain variable.

Definition 2.15 *(Liu [122]) Let ξ be an uncertain variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \ge x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \le x\}\mathrm{d}x \tag{2.120}$$

provided that at least one of the two integrals is finite.

Theorem 2.27 *(Liu [122]) Let* ξ *be an uncertain variable with uncertainty distribution* Φ*. Then*

$$E[\xi] = \int_0^{+\infty}(1-\Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x. \tag{2.121}$$

Proof: It follows from the measure inversion theorem that for almost all numbers x, we have $\mathcal{M}\{\xi \ge x\} = 1-\Phi(x)$ and $\mathcal{M}\{\xi \le x\} = \Phi(x)$. By using the definition of expected value operator, we obtain

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} \mathcal{M}\{\xi \ge x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \le x\}\mathrm{d}x \\ &= \int_0^{+\infty}(1-\Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x. \end{aligned}$$

See Figure 2.16. The theorem is proved.

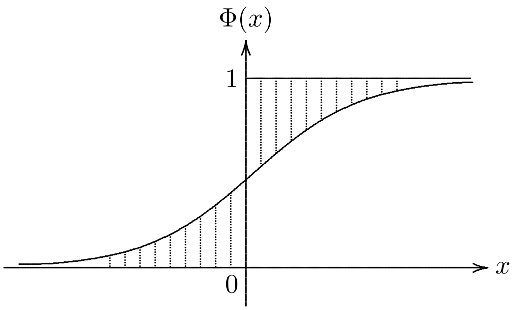

Figure 2.16: $E[\xi] = \displaystyle\int_0^{+\infty}(1-\Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x$. Reprinted from Liu [129].

Theorem 2.28 *(Liu [129]) Let* ξ *be an uncertain variable with uncertainty distribution* Φ*. Then*

$$E[\xi] = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x). \tag{2.122}$$

Proof: It follows from the integration by parts and Theorem 2.27 that the expected value is

$$\begin{aligned} E[\xi] &= \int_0^{+\infty}(1-\Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x \\ &= \int_0^{+\infty} x\mathrm{d}\Phi(x) + \int_{-\infty}^0 x\mathrm{d}\Phi(x) = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x). \end{aligned}$$

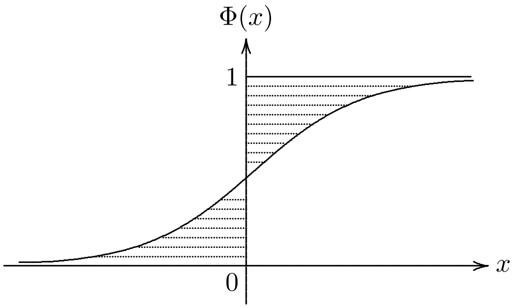

Figure 2.17: $E[\xi] = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x) = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha$. Reprinted from Liu [129].

See Figure 2.17. The theorem is proved.

Remark 2.7: If the uncertainty distribution $\Phi(x)$ has a derivative $\phi(x)$, then we immediately have

$$E[\xi] = \int_{-\infty}^{+\infty} x\phi(x)\mathrm{d}x. \tag{2.123}$$

However, it is inappropriate to regard $\phi(x)$ as an uncertainty density function because uncertain measure is not additive, i.e., generally speaking,

$$\mathcal{M}\{a \le \xi \le b\} \ne \int_a^b \phi(x)\mathrm{d}x. \tag{2.124}$$

Theorem 2.29 *(Liu [129]) Let ξ be an uncertain variable with regular uncertainty distribution Φ. Then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha. \tag{2.125}$$

Proof: Substituting $\Phi(x)$ with α and x with $\Phi^{-1}(\alpha)$, it follows from the change of variables of integral and Theorem 2.28 that the expected value is

$$E[\xi] = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x) = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha.$$

See Figure 2.17. The theorem is proved.

Exercise 2.26: Show that the linear uncertain variable $\xi \sim \mathcal{L}(a, b)$ has an expected value

$$E[\xi] = \frac{a+b}{2}. \tag{2.126}$$

Exercise 2.27: Show that the zigzag uncertain variable $\xi \sim \mathcal{Z}(a,b,c)$ has an expected value

$$E[\xi] = \frac{a+2b+c}{4}. \tag{2.127}$$

Exercise 2.28: Show that the normal uncertain variable $\xi \sim \mathcal{N}(e,\sigma)$ has an expected value e, i.e.,

$$E[\xi] = e. \tag{2.128}$$

Exercise 2.29: Show that the lognormal uncertain variable $\xi \sim \mathcal{LOGN}(e,\sigma)$ has an expected value

$$E[\xi] = \begin{cases} \sigma\sqrt{3}\exp(e)\csc(\sigma\sqrt{3}), & \text{if } \sigma < \pi/\sqrt{3} \\ +\infty, & \text{if } \sigma \ge \pi/\sqrt{3}. \end{cases} \tag{2.129}$$

This formula was first discovered by Dr. Zhongfeng Qin with the help of Maple software, and was verified again by Dr. Kai Yao through a rigorous mathematical derivation.

Exercise 2.30: Let ξ be an uncertain variable with empirical uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < x_1 \\ \alpha_i + \dfrac{(\alpha_{i+1}-\alpha_i)(x-x_i)}{x_{i+1}-x_i}, & \text{if } x_i \le x \le x_{i+1},\ 1 \le i < n \\ 1, & \text{if } x > x_n \end{cases}$$

where $x_1 < x_2 < \cdots < x_n$ and $0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le 1$. Show that

$$E[\xi] = \frac{\alpha_1+\alpha_2}{2}x_1 + \sum_{i=2}^{n-1}\frac{\alpha_{i+1}-\alpha_{i-1}}{2}x_i + \left(1 - \frac{\alpha_{n-1}+\alpha_n}{2}\right)x_n. \tag{2.130}$$

Expected Value of Monotone Function of Uncertain Variables

Theorem 2.30 *(Liu and Ha [147]) Assume $\xi_1,\xi_2,\cdots,\xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\cdots,\Phi_n$, respectively. If $f(x_1,x_2,\cdots,x_n)$ is strictly increasing with respect to $x_1,x_2,\cdots,x_m$ and strictly decreasing with respect to $x_{m+1},x_{m+2},\cdots,x_n$, then the uncertain variable $\xi = f(\xi_1,\xi_2,\cdots,\xi_n)$ has an expected value*

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha. \tag{2.131}$$

Proof: Since the function $f(x_1,x_2,\cdots,x_n)$ is strictly increasing with respect to $x_1,x_2,\cdots,x_m$ and strictly decreasing with respect to $x_{m+1},x_{m+2},\cdots,x_n$, it follows from Theorem 2.18 that the inverse uncertainty distribution of ξ is

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha),\cdots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\cdots,\Phi_n^{-1}(1-\alpha)).$$

By using Theorem 2.29, we obtain (2.131). The theorem is proved.

Exercise 2.31: Let ξ be an uncertain variable with regular uncertainty distribution Φ, and let $f(x)$ be a strictly monotone (increasing or decreasing) function. Show that

$$E[f(\xi)] = \int_0^1 f(\Phi^{-1}(\alpha))\mathrm{d}\alpha. \tag{2.132}$$

Exercise 2.32: Let ξ be an uncertain variable with uncertainty distribution Φ, and let $f(x)$ be a strictly monotone (increasing or decreasing) function. Show that

$$E[f(\xi)] = \int_{-\infty}^{+\infty} f(x)\mathrm{d}\Phi(x). \tag{2.133}$$

Exercise 2.33: Let ξ and η be independent and positive uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Show that

$$E[\xi\eta] = \int_0^1 \Phi^{-1}(\alpha)\Psi^{-1}(\alpha)\mathrm{d}\alpha. \tag{2.134}$$

Exercise 2.34: Let ξ and η be independent and positive uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Show that

$$E\left[\frac{\xi}{\eta}\right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Psi^{-1}(1-\alpha)}\mathrm{d}\alpha. \tag{2.135}$$

Exercise 2.35: Assume ξ and η are independent and positive uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Show that

$$E\left[\frac{\xi}{\xi+\eta}\right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha)+\Psi^{-1}(1-\alpha)}\mathrm{d}\alpha. \tag{2.136}$$

Linearity of Expected Value Operator

Theorem 2.31 *(Liu [129]) Let ξ and η be independent uncertain variables with finite expected values. Then for any real numbers a and b, we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \tag{2.137}$$

Proof: Without loss of generality, suppose ξ and η have regular uncertainty distributions Φ and Ψ, respectively. Otherwise, we may give the uncertainty distributions a small perturbation such that they become regular.

STEP 1: We first prove $E[a\xi] = aE[\xi]$. If $a = 0$, then the equation holds trivially. If $a > 0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha) = a\Phi^{-1}(\alpha).$$

It follows from Theorem 2.29 that

$$E[a\xi] = \int_0^1 a\Phi^{-1}(\alpha)\mathrm{d}\alpha = a\int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha = aE[\xi].$$

If $a < 0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha) = a\Phi^{-1}(1-\alpha).$$

It follows from Theorem 2.29 that

$$E[a\xi] = \int_0^1 a\Phi^{-1}(1-\alpha)\mathrm{d}\alpha = a\int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha = aE[\xi].$$

Thus we always have $E[a\xi] = aE[\xi]$.

STEP 2: We prove $E[\xi+\eta] = E[\xi] + E[\eta]$. The inverse uncertainty distribution of the sum $\xi+\eta$ is

$$\Upsilon^{-1}(\alpha) = \Phi^{-1}(\alpha) + \Psi^{-1}(\alpha).$$

It follows from Theorem 2.29 that

$$E[\xi+\eta] = \int_0^1 \Upsilon^{-1}(\alpha)\mathrm{d}\alpha = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha + \int_0^1 \Psi^{-1}(\alpha)\mathrm{d}\alpha = E[\xi] + E[\eta].$$

STEP 3: Finally, for any real numbers a and b, it follows from Steps 1 and 2 that

$$E[a\xi + b\eta] = E[a\xi] + E[b\eta] = aE[\xi] + bE[\eta].$$

The theorem is proved.

Example 2.12: Generally speaking, the expected value operator is not necessarily linear if the independence is not assumed. For example, take $(\Gamma,\mathcal{L},\mathcal{M})$ to be $\{\gamma_1,\gamma_2,\gamma_3\}$ with $\mathcal{M}\{\gamma_1\} = 0.7$, $\mathcal{M}\{\gamma_2\} = 0.3$ and $\mathcal{M}\{\gamma_3\} = 0.2$. It follows from the extension theorem that $\mathcal{M}\{\gamma_1,\gamma_2\} = 0.8$, $\mathcal{M}\{\gamma_1,\gamma_3\} = 0.7$, $\mathcal{M}\{\gamma_2,\gamma_3\} = 0.3$. Define two uncertain variables as follows,

$$\xi(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ 0, & \text{if } \gamma = \gamma_2 \\ 2, & \text{if } \gamma = \gamma_3, \end{cases} \qquad \eta(\gamma) = \begin{cases} 0, & \text{if } \gamma = \gamma_1 \\ 2, & \text{if } \gamma = \gamma_2 \\ 3, & \text{if } \gamma = \gamma_3. \end{cases}$$

Note that ξ and η are not independent, and their sum is

$$(\xi+\eta)(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ 2, & \text{if } \gamma = \gamma_2 \\ 5, & \text{if } \gamma = \gamma_3. \end{cases}$$

It is easy to verify that $E[\xi] = 0.9$, $E[\eta] = 0.8$, and $E[\xi + \eta] = 1.9$. Thus we have

$$E[\xi + \eta] > E[\xi] + E[\eta].$$

If the uncertain variables are defined by

$$\xi(\gamma) = \begin{cases} 0, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2 \\ 2, & \text{if } \gamma = \gamma_3, \end{cases} \qquad \eta(\gamma) = \begin{cases} 0, & \text{if } \gamma = \gamma_1 \\ 3, & \text{if } \gamma = \gamma_2 \\ 1, & \text{if } \gamma = \gamma_3. \end{cases}$$

Then

$$(\xi + \eta)(\gamma) = \begin{cases} 0, & \text{if } \gamma = \gamma_1 \\ 4, & \text{if } \gamma = \gamma_2 \\ 3, & \text{if } \gamma = \gamma_3. \end{cases}$$

It is easy to verify that $E[\xi] = 0.5$, $E[\eta] = 0.9$, and $E[\xi + \eta] = 1.2$. Thus we have

$$E[\xi + \eta] < E[\xi] + E[\eta].$$

Comonotonic Functions of Uncertain Variable

Two real-valued functions f and g are said to be *comonotonic* if for any numbers x and y, we always have

$$(f(x) - f(y))(g(x) - g(y)) \geq 0. \tag{2.138}$$

It is easy to verify that (i) any function is comonotonic with any positive constant multiple of the function; (ii) any monotone increasing functions are comonotonic with each other; and (iii) any monotone decreasing functions are also comonotonic with each other.

Theorem 2.32 *(Yang [240]) Let f and g be comonotonic functions. Then for any uncertain variable ξ, we have*

$$E[f(\xi) + g(\xi)] = E[f(\xi)] + E[g(\xi)]. \tag{2.139}$$

Proof: Without loss of generality, suppose $f(\xi)$ and $g(\xi)$ have regular uncertainty distributions Φ and Ψ, respectively. Otherwise, we may give the uncertainty distributions a small perturbation such that they become regular. Since f and g are comonotonic functions, at least one of the following relations is true,

$$\{f(\xi) \leq \Phi^{-1}(\alpha)\} \subset \{g(\xi) \leq \Psi^{-1}(\alpha)\},$$

$$\{f(\xi) \leq \Phi^{-1}(\alpha)\} \supset \{g(\xi) \leq \Psi^{-1}(\alpha)\}.$$

On the one hand, we have

$$\begin{aligned}
&\mathcal{M}\{f(\xi)+g(\xi)\le\Phi^{-1}(\alpha)+\Psi^{-1}(\alpha)\}\\
&\ge\mathcal{M}\{(f(\xi)\le\Phi^{-1}(\alpha))\cap(g(\xi)\le\Psi^{-1}(\alpha))\}\\
&=\mathcal{M}\{f(\xi)\le\Phi^{-1}(\alpha)\}\wedge\mathcal{M}\{g(\xi)\le\Psi^{-1}(\alpha)\}\\
&=\alpha\wedge\alpha=\alpha.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\mathcal{M}\{f(\xi)+g(\xi)\le\Phi^{-1}(\alpha)+\Psi^{-1}(\alpha)\}\\
&\le\mathcal{M}\{(f(\xi)\le\Phi^{-1}(\alpha))\cup(g(\xi)\le\Psi^{-1}(\alpha))\}\\
&=\mathcal{M}\{f(\xi)\le\Phi^{-1}(\alpha)\}\vee\mathcal{M}\{g(\xi)\le\Psi^{-1}(\alpha)\}\\
&=\alpha\vee\alpha=\alpha.
\end{aligned}$$

It follows that

$$\mathcal{M}\{f(\xi)+g(\xi)\le\Phi^{-1}(\alpha)+\Psi^{-1}(\alpha)\}=\alpha$$

holds for each α. That is, $\Phi^{-1}(\alpha)+\Psi^{-1}(\alpha)$ is the inverse uncertainty distribution of $f(\xi)+g(\xi)$. By using Theorem 2.29, we obtain

$$\begin{aligned}
E[f(\xi)+g(\xi)]&=\int_0^1(\Phi^{-1}(\alpha)+\Psi^{-1}(\alpha))\mathrm{d}\alpha\\
&=\int_0^1\Phi^{-1}(\alpha)\mathrm{d}\alpha+\int_0^1\Psi^{-1}(\alpha)\mathrm{d}\alpha\\
&=E[f(\xi)]+E[g(\xi)].
\end{aligned}$$

The theorem is verified.

Exercise 2.36: Let ξ be a positive uncertain variable. Show that $\ln x$ and $\exp(x)$ are comonotonic functions on $(0,+\infty)$, and

$$E[\ln\xi+\exp(\xi)]=E[\ln\xi]+E[\exp(\xi)]. \tag{2.140}$$

Exercise 2.37: Let ξ be a positive uncertain variable. Show that $x, x^2, \cdots, x^n$ are comonotonic functions on $[0,+\infty)$, and

$$E[\xi+\xi^2+\cdots+\xi^n]=E[\xi]+E[\xi^2]+\cdots+E[\xi^n]. \tag{2.141}$$

Some Inequalities

Theorem 2.33 *(Liu [122]) Let ξ be an uncertain variable, and let f be a nonnegative function. If f is even and increasing on $[0,\infty)$, then for any given number $t>0$, we have*

$$\mathcal{M}\{|\xi|\ge t\}\le\frac{E[f(\xi)]}{f(t)}. \tag{2.142}$$

Proof: It is clear that $\mathcal{M}\{|\xi| \ge f^{-1}(r)\}$ is a monotone decreasing function of r on $[0, \infty)$. It follows from the nonnegativity of $f(\xi)$ that

$$\begin{aligned} E[f(\xi)] &= \int_0^{+\infty} \mathcal{M}\{f(\xi) \ge x\}\mathrm{d}x = \int_0^{+\infty} \mathcal{M}\{|\xi| \ge f^{-1}(x)\}\mathrm{d}x \\ &\ge \int_0^{f(t)} \mathcal{M}\{|\xi| \ge f^{-1}(x)\}\mathrm{d}x \ge \int_0^{f(t)} \mathcal{M}\{|\xi| \ge f^{-1}(f(t))\}\mathrm{d}x \\ &= \int_0^{f(t)} \mathcal{M}\{|\xi| \ge t\}\mathrm{d}x = f(t) \cdot \mathcal{M}\{|\xi| \ge t\} \end{aligned}$$

which proves the inequality.

Theorem 2.34 *(Liu [122], Markov Inequality) Let ξ be an uncertain variable. Then for any given numbers $t > 0$ and $p > 0$, we have*

$$\mathcal{M}\{|\xi| \ge t\} \le \frac{E[|\xi|^p]}{t^p}. \tag{2.143}$$

Proof: It is a special case of Theorem 2.33 when $f(x) = |x|^p$.

Example 2.13: For any given positive number t, we define an uncertain variable as follows,

$$\xi = \begin{cases} 0 \text{ with uncertain measure } 1/2 \\ t \text{ with uncertain measure } 1/2. \end{cases}$$

Then $E[\xi^p] = t^p/2$ and $\mathcal{M}\{\xi \ge t\} = 1/2 = E[\xi^p]/t^p$.

Theorem 2.35 *(Liu [122], Hölder's Inequality) Let p and q be positive numbers with $1/p + 1/q = 1$, and let ξ and η be independent uncertain variables with $E[|\xi|^p] < \infty$ and $E[|\eta|^q] < \infty$. Then we have*

$$E[|\xi\eta|] \le \sqrt[p]{E[|\xi|^p]}\sqrt[q]{E[|\eta|^q]}. \tag{2.144}$$

Proof: The inequality holds trivially if at least one of ξ and η is zero a.s. Now we assume $E[|\xi|^p] > 0$ and $E[|\eta|^q] > 0$. It is easy to prove that the function $f(x, y) = \sqrt[p]{x}\sqrt[q]{y}$ is a concave function on $\{(x, y) : x \ge 0, y \ge 0\}$. Thus for any point (x_0, y_0) with $x_0 > 0$ and $y_0 > 0$, there exist two real numbers a and b such that

$$f(x, y) - f(x_0, y_0) \le a(x - x_0) + b(y - y_0), \quad \forall x \ge 0, y \ge 0.$$

Letting $x_0 = E[|\xi|^p]$, $y_0 = E[|\eta|^q]$, $x = |\xi|^p$ and $y = |\eta|^q$, we have

$$f(|\xi|^p, |\eta|^q) - f(E[|\xi|^p], E[|\eta|^q]) \le a(|\xi|^p - E[|\xi|^p]) + b(|\eta|^q - E[|\eta|^q]).$$

Taking the expected values on both sides, we obtain

$$E[f(|\xi|^p, |\eta|^q)] \le f(E[|\xi|^p], E[|\eta|^q]).$$

Hence the inequality (2.144) holds.

Theorem 2.36 *(Liu [122], Minkowski Inequality) Let p be a real number with $p \ge 1$, and let ξ and η be independent uncertain variables with $E[|\xi|^p] < \infty$ and $E[|\eta|^p] < \infty$. Then we have*

$$\sqrt[p]{E[|\xi+\eta|^p]} \le \sqrt[p]{E[|\xi|^p]} + \sqrt[p]{E[|\eta|^p]}. \tag{2.145}$$

Proof: The inequality holds trivially if at least one of ξ and η is zero a.s. Now we assume $E[|\xi|^p] > 0$ and $E[|\eta|^p] > 0$. It is easy to prove that the function $f(x,y) = (\sqrt[p]{x} + \sqrt[p]{y})^p$ is a concave function on $\{(x,y) : x \ge 0, y \ge 0\}$. Thus for any point (x_0, y_0) with $x_0 > 0$ and $y_0 > 0$, there exist two real numbers a and b such that

$$f(x,y) - f(x_0,y_0) \le a(x - x_0) + b(y - y_0), \quad \forall x \ge 0, y \ge 0.$$

Letting $x_0 = E[|\xi|^p]$, $y_0 = E[|\eta|^p]$, $x = |\xi|^p$ and $y = |\eta|^p$, we have

$$f(|\xi|^p, |\eta|^p) - f(E[|\xi|^p], E[|\eta|^p]) \le a(|\xi|^p - E[|\xi|^p]) + b(|\eta|^p - E[|\eta|^p]).$$

Taking the expected values on both sides, we obtain

$$E[f(|\xi|^p, |\eta|^p)] \le f(E[|\xi|^p], E[|\eta|^p]).$$

Hence the inequality (2.145) holds.

Theorem 2.37 *(Liu [122], Jensen's Inequality) Let ξ be an uncertain variable, and let f be a convex function. If $E[\xi]$ and $E[f(\xi)]$ are finite, then*

$$f(E[\xi]) \le E[f(\xi)]. \tag{2.146}$$

Especially, when $f(x) = |x|^p$ and $p \ge 1$, we have $|E[\xi]|^p \le E[|\xi|^p]$.

Proof: Since f is a convex function, for each y, there exists a number k such that $f(x) - f(y) \ge k \cdot (x - y)$. Replacing x with ξ and y with $E[\xi]$, we obtain

$$f(\xi) - f(E[\xi]) \ge k \cdot (\xi - E[\xi]).$$

Taking the expected values on both sides, we have

$$E[f(\xi)] - f(E[\xi]) \ge k \cdot (E[\xi] - E[\xi]) = 0$$

which proves the inequality.

Exercise 2.38: (Zhang [268]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with finite expected values, and let f be a convex function. Show that

$$f(E[\xi_1], E[\xi_2], \cdots, E[\xi_n]) \le E[f(\xi_1, \xi_2, \cdots, \xi_n)]. \tag{2.147}$$

2.6 Variance

The variance of uncertain variable provides a degree of the spread of the distribution around its expected value. A small value of variance indicates that the uncertain variable is tightly concentrated around its expected value; and a large value of variance indicates that the uncertain variable has a wide spread around its expected value.

Definition 2.16 *(Liu [122]) Let ξ be an uncertain variable with finite expected value e. Then the variance of ξ is*

$$V[\xi]=E[(\xi-e)^2]. \tag{2.148}$$

This definition tells us that the variance is just the expected value of $(\xi-e)^2$. Since $(\xi-e)^2$ is a nonnegative uncertain variable, we also have

$$V[\xi]=\int_0^{+\infty}\mathcal{M}\{(\xi-e)^2\ge x\}\mathrm{d}x. \tag{2.149}$$

Theorem 2.38 *If ξ is an uncertain variable with finite expected value, a and b are real numbers, then*

$$V[a\xi+b]=a^2V[\xi]. \tag{2.150}$$

Proof: Let e be the expected value of ξ. Then $a\xi+b$ has an expected value $ae+b$. It follows from the definition of variance that

$$V[a\xi+b]=E\left[(a\xi+b-(ae+b))^2\right]=a^2E[(\xi-e)^2]=a^2V[\xi].$$

The theorem is thus verified.

Theorem 2.39 *Let ξ be an uncertain variable with expected value e. Then $V[\xi]=0$ if and only if $\mathcal{M}\{\xi=e\}=1$. That is, the uncertain variable ξ is essentially the constant e.*

Proof: We first assume $V[\xi]=0$. It follows from the equation (2.149) that

$$\int_0^{+\infty}\mathcal{M}\{(\xi-e)^2\ge x\}\mathrm{d}x=0$$

which implies $\mathcal{M}\{(\xi-e)^2\ge x\}=0$ for any $x>0$. Hence we have

$$\mathcal{M}\{(\xi-e)^2=0\}=1.$$

That is, $\mathcal{M}\{\xi=e\}=1$. Conversely, assume $\mathcal{M}\{\xi=e\}=1$. Then we immediately have $\mathcal{M}\{(\xi-e)^2=0\}=1$ and $\mathcal{M}\{(\xi-e)^2\ge x\}=0$ for any $x>0$. Thus

$$V[\xi]=\int_0^{+\infty}\mathcal{M}\{(\xi-e)^2\ge x\}\mathrm{d}x=0.$$

The theorem is proved.

Theorem 2.40 *(Yao [254]) Let ξ and η be independent uncertain variables whose variances exist. Then*

$$\sqrt{V[\xi+\eta]} \le \sqrt{V[\xi]} + \sqrt{V[\eta]}. \tag{2.151}$$

Proof: It is a special case of Theorem 2.36 when $p = 2$ and the uncertain variables ξ and η are replaced with $\xi - E[\xi]$ and $\eta - E[\eta]$, respectively.

Theorem 2.41 *(Liu [122], Chebyshev Inequality) Let ξ be an uncertain variable whose variance exists. Then for any given number $t > 0$, we have*

$$\mathcal{M}\{|\xi - E[\xi]| \ge t\} \le \frac{V[\xi]}{t^2}. \tag{2.152}$$

Proof: It is a special case of Theorem 2.33 when the uncertain variable ξ is replaced with $\xi - E[\xi]$, and $f(x) = x^2$.

Example 2.14: For any given positive number t, we define an uncertain variable as follows,

$$\xi = \begin{cases} -t \text{ with uncertain measure } 1/2 \\ t \text{ with uncertain measure } 1/2. \end{cases}$$

Then $V[\xi] = t^2$ and $\mathcal{M}\{|\xi - E[\xi]| \ge t\} = 1 = V[\xi]/t^2$.

How to Obtain Variance from Uncertainty Distribution?

Let ξ be an uncertain variable with expected value e. If we only know its uncertainty distribution Φ, then the variance

$$\begin{aligned} V[\xi] &= \int_0^{+\infty} \mathcal{M}\{(\xi - e)^2 \ge x\}\mathrm{d}x \\ &= \int_0^{+\infty} \mathcal{M}\{(\xi \ge e + \sqrt{x}) \cup (\xi \le e - \sqrt{x})\}\mathrm{d}x \\ &\le \int_0^{+\infty} (\mathcal{M}\{\xi \ge e + \sqrt{x}\} + \mathcal{M}\{\xi \le e - \sqrt{x}\})\mathrm{d}x \\ &= \int_0^{+\infty} (1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x}))\mathrm{d}x. \end{aligned}$$

Thus we have the following stipulation.

Stipulation 2.3 *Let ξ be an uncertain variable with uncertainty distribution Φ and finite expected value e. Then*

$$V[\xi] = \int_0^{+\infty} (1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x}))\mathrm{d}x. \tag{2.153}$$

Theorem 2.42 *Let ξ be an uncertain variable with uncertainty distribution Φ and finite expected value e. Then*

$$V[\xi] = \int_{-\infty}^{+\infty} (x-e)^2 \mathrm{d}\Phi(x). \tag{2.154}$$

Proof: This theorem is based on Stipulation 2.3 that says the variance of ξ is

$$V[\xi] = \int_0^{+\infty} (1-\Phi(e+\sqrt{y}))\mathrm{d}y + \int_0^{+\infty} \Phi(e-\sqrt{y})\mathrm{d}y.$$

Substituting $e+\sqrt{y}$ with x and y with $(x-e)^2$, the change of variables and integration by parts produce

$$\int_0^{+\infty} (1-\Phi(e+\sqrt{y}))\mathrm{d}y = \int_e^{+\infty} (1-\Phi(x))\mathrm{d}(x-e)^2 = \int_e^{+\infty} (x-e)^2\mathrm{d}\Phi(x).$$

Similarly, substituting $e-\sqrt{y}$ with x and y with $(x-e)^2$, we obtain

$$\int_0^{+\infty} \Phi(e-\sqrt{y})\mathrm{d}y = \int_e^{-\infty} \Phi(x)\mathrm{d}(x-e)^2 = \int_{-\infty}^{e} (x-e)^2\mathrm{d}\Phi(x).$$

It follows that the variance is

$$V[\xi] = \int_e^{+\infty} (x-e)^2\mathrm{d}\Phi(x) + \int_{-\infty}^{e} (x-e)^2\mathrm{d}\Phi(x) = \int_{-\infty}^{+\infty} (x-e)^2\mathrm{d}\Phi(x).$$

The theorem is verified.

Theorem 2.43 *(Yao [254]) Let ξ be an uncertain variable with regular uncertainty distribution Φ and finite expected value e. Then*

$$V[\xi] = \int_0^1 (\Phi^{-1}(\alpha)-e)^2\mathrm{d}\alpha. \tag{2.155}$$

Proof: Substituting $\Phi(x)$ with α and x with $\Phi^{-1}(\alpha)$, it follows from the change of variables of integral and Theorem 2.42 that the variance is

$$V[\xi] = \int_{-\infty}^{+\infty} (x-e)^2\mathrm{d}\Phi(x) = \int_0^1 (\Phi^{-1}(\alpha)-e)^2\mathrm{d}\alpha.$$

The theorem is verified.

Exercise 2.39: Show that the linear uncertain variable $\xi \sim \mathcal{L}(a,b)$ has a variance

$$V[\xi] = \frac{(b-a)^2}{12}. \tag{2.156}$$

Exercise 2.40: Show that the normal uncertain variable $\xi \sim \mathcal{N}(e,\sigma)$ has a variance

$$V[\xi] = \sigma^2. \tag{2.157}$$

Theorem 2.44 *(Yao [254]) Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then the uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has a variance*

$$V[\xi] = \int_0^1 (f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) - e)^2 \mathrm{d}\alpha$$

where e is the expected value of ξ.

Proof: Since the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, the inverse uncertainty distribution of ξ is

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

It follows from Theorem 2.43 that the result holds.

Exercise 2.41: Let ξ and η be independent uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Assume there exist two real numbers a and b such that

$$\Phi^{-1}(\alpha) = a\Psi^{-1}(\alpha) + b \tag{2.158}$$

for all $\alpha \in (0, 1)$. Show that

$$\sqrt{V[\xi + \eta]} = \sqrt{V[\xi]} + \sqrt{V[\eta]} \tag{2.159}$$

in the sense of Stipulation 2.3.

Remark 2.8: If ξ and η are independent linear uncertain variables, then the condition (2.158) is met. If they are independent normal uncertain variables, then the condition (2.158) is also met.

2.7 Moment

Definition 2.17 *(Liu [122]) Let ξ be an uncertain variable and let k be a positive integer. Then $E[\xi^k]$ is called the k-th moment of ξ.*

Theorem 2.45 *Let ξ be an uncertain variable with uncertainty distribution Φ, and let k be an odd number. Then the k-th moment of ξ is*

$$E[\xi^k] = \int_0^{+\infty} (1 - \Phi(\sqrt[k]{x}))\mathrm{d}x - \int_{-\infty}^0 \Phi(\sqrt[k]{x})\mathrm{d}x. \tag{2.160}$$

Proof: Since k is an odd number, it follows from the definition of expected value operator that

$$\begin{aligned}E[\xi^k] &= \int_0^{+\infty} \mathcal{M}\{\xi^k \geq x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi^k \leq x\}\mathrm{d}x \\ &= \int_0^{+\infty} \mathcal{M}\{\xi \geq \sqrt[k]{x}\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \leq \sqrt[k]{x}\}\mathrm{d}x \\ &= \int_0^{+\infty} (1 - \Phi(\sqrt[k]{x}))\mathrm{d}x - \int_{-\infty}^0 \Phi(\sqrt[k]{x})\mathrm{d}x.\end{aligned}$$

The theorem is proved.

However, when k is an even number, the k-th moment of ξ cannot be uniquely determined by the uncertainty distribution Φ. In this case, we have

$$\begin{aligned}E[\xi^k] &= \int_0^{+\infty} \mathcal{M}\{\xi^k \geq x\}\mathrm{d}x \\ &= \int_0^{+\infty} \mathcal{M}\{(\xi \geq \sqrt[k]{x}) \cup (\xi \leq -\sqrt[k]{x})\}\mathrm{d}x \\ &\leq \int_0^{+\infty} (\mathcal{M}\{\xi \geq \sqrt[k]{x}\} + \mathcal{M}\{\xi \leq -\sqrt[k]{x}\})\mathrm{d}x \\ &= \int_0^{+\infty} (1 - \Phi(\sqrt[k]{x}) + \Phi(-\sqrt[k]{x}))\mathrm{d}x.\end{aligned}$$

Thus for the even number k, we have the following stipulation.

Stipulation 2.4 *Let ξ be an uncertain variable with uncertainty distribution Φ, and let k be an even number. Then the k-th moment of ξ is*

$$E[\xi^k] = \int_0^{+\infty} (1 - \Phi(\sqrt[k]{x}) + \Phi(-\sqrt[k]{x}))\mathrm{d}x. \tag{2.161}$$

Theorem 2.46 *Let ξ be an uncertain variable with uncertainty distribution Φ, and let k be a positive integer. Then the k-th moment of ξ is*

$$E[\xi^k] = \int_{-\infty}^{+\infty} x^k \mathrm{d}\Phi(x). \tag{2.162}$$

Proof: When k is an odd number, Theorem 2.45 says that the k-th moment is

$$E[\xi^k] = \int_0^{+\infty} (1 - \Phi(\sqrt[k]{y}))\mathrm{d}y - \int_{-\infty}^0 \Phi(\sqrt[k]{y})\mathrm{d}y.$$

Substituting $\sqrt[k]{y}$ with x and y with x^k, the change of variables and integration by parts produce

$$\int_0^{+\infty} (1 - \Phi(\sqrt[k]{y}))\mathrm{d}y = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x^k = \int_0^{+\infty} x^k \mathrm{d}\Phi(x)$$

and

$$\int_{-\infty}^{0} \Phi(\sqrt[k]{y})\mathrm{d}y = \int_{-\infty}^{0} \Phi(x)\mathrm{d}x^k = -\int_{-\infty}^{0} x^k \mathrm{d}\Phi(x).$$

Thus we have

$$E[\xi^k] = \int_0^{+\infty} x^k \mathrm{d}\Phi(x) + \int_{-\infty}^{0} x^k \mathrm{d}\Phi(x) = \int_{-\infty}^{+\infty} x^k \mathrm{d}\Phi(x).$$

When k is an even number, the theorem is based on Stipulation 2.4 that says the k-th moment is

$$E[\xi^k] = \int_0^{+\infty} (1 - \Phi(\sqrt[k]{y}) + \Phi(-\sqrt[k]{y}))\mathrm{d}y.$$

Substituting $\sqrt[k]{y}$ with x and y with x^k, the change of variables and integration by parts produce

$$\int_0^{+\infty} (1 - \Phi(\sqrt[k]{y}))\mathrm{d}y = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x^k = \int_0^{+\infty} x^k \mathrm{d}\Phi(x).$$

Similarly, substituting $-\sqrt[k]{y}$ with x and y with x^k, we obtain

$$\int_0^{+\infty} \Phi(-\sqrt[k]{y})\mathrm{d}y = \int_{-\infty}^{0} \Phi(x)\mathrm{d}x^k = \int_{-\infty}^{0} x^k \mathrm{d}\Phi(x).$$

It follows that the k-th moment is

$$E[\xi^k] = \int_0^{+\infty} x^k \mathrm{d}\Phi(x) + \int_{-\infty}^{0} x^k \mathrm{d}\Phi(x) = \int_{-\infty}^{+\infty} x^k \mathrm{d}\Phi(x).$$

The theorem is thus verified for any positive integer k.

Theorem 2.47 *(Sheng and Kar [213]) Let ξ be an uncertain variable with regular uncertainty distribution Φ, and let k be a positive integer. Then the k-th moment of ξ is*

$$E[\xi^k] = \int_0^1 (\Phi^{-1}(\alpha))^k \mathrm{d}\alpha. \tag{2.163}$$

Proof: Substituting $\Phi(x)$ with α and x with $\Phi^{-1}(\alpha)$, it follows from the change of variables of integral and Theorem 2.46 that the k-th moment is

$$E[\xi^k] = \int_{-\infty}^{+\infty} x^k \mathrm{d}\Phi(x) = \int_0^1 (\Phi^{-1}(\alpha))^k \mathrm{d}\alpha.$$

The theorem is verified.

Exercise 2.42: Show that the second moment of linear uncertain variable $\xi \sim \mathcal{L}(a, b)$ is

$$E[\xi^2] = \frac{a^2 + ab + b^2}{3}. \tag{2.164}$$

Exercise 2.43: Show that the second moment of normal uncertain variable $\xi \sim \mathcal{N}(e,\sigma)$ is
$$E[\xi^2] = e^2 + \sigma^2. \tag{2.165}$$

Theorem 2.48 *(Sheng and Kar [213]) Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, and k is a positive integer. If $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then the k-th moment of $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is*
$$E[\xi^k] = \int_0^1 f^k(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha.$$

Proof: Since the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, the inverse uncertainty distribution of ξ is
$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$
It follows from Theorem 2.47 that the result holds.

2.8 Entropy

This section provides a definition of entropy to characterize the uncertainty of uncertain variables.

Definition 2.18 *(Liu [125]) Suppose that ξ is an uncertain variable with uncertainty distribution Φ. Then its entropy is defined by*
$$H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x))\mathrm{d}x \tag{2.166}$$
where $S(t) = -t\ln t - (1-t)\ln(1-t)$.

Example 2.15: Let ξ be an uncertain variable with uncertainty distribution
$$\Phi(x) = \begin{cases} 0, & \text{if } x < a \\ 1, & \text{if } x \ge a. \end{cases} \tag{2.167}$$
Essentially, ξ is a constant a. It follows from the definition of entropy that
$$H[\xi] = -\int_{-\infty}^{a} (0\ln 0 + 1\ln 1)\,\mathrm{d}x - \int_a^{+\infty} (1\ln 1 + 0\ln 0)\,\mathrm{d}x = 0.$$
This means a constant has no uncertainty.

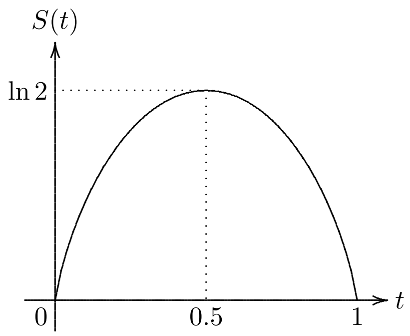

Figure 2.18: Function $S(t) = -t\ln t - (1-t)\ln(1-t)$. It is easy to verify that $S(t)$ is a symmetric function about $t = 0.5$, strictly increasing on the interval $[0, 0.5]$, strictly decreasing on the interval $[0.5, 1]$, and reaches its unique maximum $\ln 2$ at $t = 0.5$. Reprinted from Liu [129].

Example 2.16: Let ξ be a linear uncertain variable $\mathcal{L}(a, b)$. Then its entropy is

$$H[\xi] = -\int_a^b \left(\frac{x-a}{b-a}\ln\frac{x-a}{b-a} + \frac{b-x}{b-a}\ln\frac{b-x}{b-a}\right) \mathrm{d}x = \frac{b-a}{2}. \tag{2.168}$$

Exercise 2.44: Show that the zigzag uncertain variable $\xi \sim \mathcal{Z}(a, b, c)$ has an entropy

$$H[\xi] = \frac{c-a}{2}. \tag{2.169}$$

Exercise 2.45: Show that the normal uncertain variable $\xi \sim \mathcal{N}(e, \sigma)$ has an entropy

$$H[\xi] = \frac{\pi\sigma}{\sqrt{3}}. \tag{2.170}$$

Theorem 2.49 *Let ξ be an uncertain variable. Then $H[\xi] \ge 0$ and equality holds if ξ is essentially a constant.*

Proof: The nonnegativity is clear. In addition, when an uncertain variable tends to a constant, its entropy tends to the minimum 0.

Theorem 2.50 *Let ξ be an uncertain variable taking values on the interval $[a, b]$. Then*

$$H[\xi] \le (b-a)\ln 2 \tag{2.171}$$

and equality holds if ξ has an uncertainty distribution $\Phi(x) = 0.5$ on $[a, b]$.

Proof: The theorem follows from the fact that the function $S(t)$ reaches its maximum $\ln 2$ at $t = 0.5$.

Theorem 2.51 *Let ξ be an uncertain variable, and let c be a real number. Then*

$$H[\xi + c] = H[\xi]. \tag{2.172}$$

That is, the entropy is invariant under arbitrary translations.

Proof: Write the uncertainty distribution of ξ by Φ. Then the uncertain variable $\xi + c$ has an uncertainty distribution $\Phi(x - c)$. It follows from the definition of entropy that

$$H[\xi + c] = \int_{-\infty}^{+\infty} S\left(\Phi(x - c)\right) \mathrm{d}x = \int_{-\infty}^{+\infty} S(\Phi(x))\mathrm{d}x = H[\xi].$$

The theorem is proved.

Theorem 2.52 *(Dai and Chen [27]) Let ξ be an uncertain variable with regular uncertainty distribution Φ. Then*

$$H[\xi] = \int_0^1 \Phi^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha. \tag{2.173}$$

Proof: It is clear that $S(\alpha)$ is a derivable function with $S'(\alpha) = -\ln\alpha/(1-\alpha)$. Since

$$S(\Phi(x)) = \int_0^{\Phi(x)} S'(\alpha)\mathrm{d}\alpha = -\int_{\Phi(x)}^1 S'(\alpha)\mathrm{d}\alpha,$$

we have

$$H[\xi] = \int_{-\infty}^{+\infty} S(\Phi(x))\mathrm{d}x = \int_{-\infty}^{0}\int_0^{\Phi(x)} S'(\alpha)\mathrm{d}\alpha\mathrm{d}x - \int_0^{+\infty}\int_{\Phi(x)}^1 S'(\alpha)\mathrm{d}\alpha\mathrm{d}x.$$

It follows from Fubini theorem that

$$\begin{aligned} H[\xi] &= \int_0^{\Phi(0)}\int_{\Phi^{-1}(\alpha)}^0 S'(\alpha)\mathrm{d}x\mathrm{d}\alpha - \int_{\Phi(0)}^1\int_0^{\Phi^{-1}(\alpha)} S'(\alpha)\mathrm{d}x\mathrm{d}\alpha \\ &= -\int_0^{\Phi(0)} \Phi^{-1}(\alpha)S'(\alpha)\mathrm{d}\alpha - \int_{\Phi(0)}^1 \Phi^{-1}(\alpha)S'(\alpha)\mathrm{d}\alpha \\ &= -\int_0^1 \Phi^{-1}(\alpha)S'(\alpha)\mathrm{d}\alpha = \int_0^1 \Phi^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha. \end{aligned}$$

The theorem is verified.

Theorem 2.53 *(Dai and Chen [27]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then the uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an entropy*

$$H[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) \ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha.$$

Proof: Since $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, it follows from Theorem 2.18 that the inverse uncertainty distribution of ξ is

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

By using Theorem 2.52, we get the entropy formula.

Exercise 2.46: Let ξ and η be independent and positive uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Show that

$$H[\xi\eta] = \int_0^1 \Phi^{-1}(\alpha)\Psi^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha.$$

Exercise 2.47: Let ξ and η be independent and positive uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Show that

$$H\left[\frac{\xi}{\eta}\right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Psi^{-1}(1-\alpha)} \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha.$$

Exercise 2.48: Let ξ and η be independent and positive uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Show that

$$H\left[\frac{\xi}{\xi+\eta}\right] = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha)+\Psi^{-1}(1-\alpha)} \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha.$$

Theorem 2.54 *(Dai and Chen [27]) Let ξ and η be independent uncertain variables. Then for any real numbers a and b, we have*

$$H[a\xi + b\eta] = |a|H[\xi] + |b|H[\eta]. \tag{2.174}$$

Proof: Without loss of generality, suppose ξ and η have regular uncertainty distributions Φ and Ψ, respectively. Otherwise, we may give the uncertainty distributions a small perturbation such that they become regular.

STEP 1: We prove $H[a\xi] = |a|H[\xi]$. If $a > 0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha) = a\Phi^{-1}(\alpha).$$

It follows from Theorem 2.52 that

$$H[a\xi] = \int_0^1 a\Phi^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha = a\int_0^1 \Phi^{-1}(\alpha) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha = |a|H[\xi].$$

If $a = 0$, then we immediately have $H[a\xi] = 0 = |a|H[\xi]$. If $a < 0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha) = a\Phi^{-1}(1-\alpha).$$

It follows from Theorem 2.52 that

$$H[a\xi] = \int_0^1 a\Phi^{-1}(1-\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha = (-a)\int_0^1 \Phi^{-1}(\alpha)\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha = |a|H[\xi].$$

Thus we always have $H[a\xi] = |a|H[\xi]$.

Step 2: We prove $H[\xi+\eta] = H[\xi] + H[\eta]$. Note that the inverse uncertainty distribution of $\xi+\eta$ is

$$\Upsilon^{-1}(\alpha) = \Phi^{-1}(\alpha) + \Psi^{-1}(\alpha).$$

It follows from Theorem 2.52 that

$$H[\xi+\eta] = \int_0^1 (\Phi^{-1}(\alpha) + \Psi^{-1}(\alpha))\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha = H[\xi] + H[\eta].$$

Step 3: Finally, for any real numbers a and b, it follows from Steps 1 and 2 that

$$H[a\xi + b\eta] = H[a\xi] + H[b\eta] = |a|H[\xi] + |b|H[\eta].$$

The theorem is proved.

Maximum Entropy Principle

Given some constraints, for example, expected value and variance, there are usually multiple compatible uncertainty distributions. Which uncertainty distribution shall we take? The *maximum entropy principle* attempts to select the uncertainty distribution that has maximum entropy and satisfies the prescribed constraints.

Theorem 2.55 *(Chen and Dai [15]) Let ξ be an uncertain variable whose uncertainty distribution is arbitrary but the expected value e and variance σ^2. Then*

$$H[\xi] \le \frac{\pi\sigma}{\sqrt{3}} \tag{2.175}$$

and the equality holds if ξ is a normal uncertain variable $\mathcal{N}(e,\sigma)$.

Proof: Let $\Phi(x)$ be the uncertainty distribution of ξ and write $\Psi(x) = \Phi(2e-x)$ for $x \ge e$. It follows from the stipulation (2.3) and the change of variable of integral that the variance is

$$V[\xi] = 2\int_e^{+\infty}(x-e)(1-\Phi(x))\mathrm{d}x + 2\int_e^{+\infty}(x-e)\Psi(x)\mathrm{d}x = \sigma^2.$$

Thus there exists a real number κ such that

$$2\int_e^{+\infty}(x-e)(1-\Phi(x))\mathrm{d}x = \kappa\sigma^2,$$

$$2\int_{e}^{+\infty}(x-e)\Psi(x)\mathrm{d}x=(1-\kappa)\sigma^2.$$

The maximum entropy distribution Φ should maximize the entropy

$$H[\xi]=\int_{-\infty}^{+\infty}S(\Phi(x))\mathrm{d}x=\int_{e}^{+\infty}S(\Phi(x))\mathrm{d}x+\int_{e}^{+\infty}S(\Psi(x))\mathrm{d}x$$

subject to the above two constraints. The Lagrangian is

$$\begin{aligned}L=&\int_{e}^{+\infty}S(\Phi(x))\mathrm{d}x+\int_{e}^{+\infty}S(\Psi(x))\mathrm{d}x\\&-\alpha\left(2\int_{e}^{+\infty}(x-e)(1-\Phi(x))\mathrm{d}x-\kappa\sigma^2\right)\\&-\beta\left(2\int_{e}^{+\infty}(x-e)\Psi(x)\mathrm{d}x-(1-\kappa)\sigma^2\right).\end{aligned}$$

The maximum entropy distribution meets Euler-Lagrange equations

$$\ln\Phi(x)-\ln(1-\Phi(x))=2\alpha(x-e),$$

$$\ln\Psi(x)-\ln(1-\Psi(x))=2\beta(e-x).$$

Thus Φ and Ψ have the forms

$$\Phi(x)=(1+\exp(2\alpha(e-x)))^{-1},$$

$$\Psi(x)=(1+\exp(2\beta(x-e)))^{-1}.$$

Substituting them into the variance constraints, we get

$$\Phi(x)=\left(1+\exp\left(\frac{\pi(e-x)}{\sqrt{6\kappa}\sigma}\right)\right)^{-1},$$

$$\Psi(x)=\left(1+\exp\left(\frac{\pi(x-e)}{\sqrt{6(1-\kappa)}\sigma}\right)\right)^{-1}.$$

Then the entropy is

$$H[\xi]=\frac{\pi\sigma\sqrt{\kappa}}{\sqrt{6}}+\frac{\pi\sigma\sqrt{1-\kappa}}{\sqrt{6}}$$

which achieves the maximum when $\kappa=1/2$. Thus the maximum entropy distribution is just the normal uncertainty distribution $\mathcal{N}(e,\sigma)$.

2.9 Distance

Definition 2.19 *(Liu [122]) The distance between uncertain variables ξ and η is defined as*

$$d(\xi,\eta) = E[|\xi-\eta|]. \tag{2.176}$$

That is, the distance between ξ and η is just the expected value of $|\xi-\eta|$. Since $|\xi-\eta|$ is a nonnegative uncertain variable, we always have

$$d(\xi,\eta) = \int_0^{+\infty} \mathcal{M}\{|\xi-\eta| \ge x\}\mathrm{d}x. \tag{2.177}$$

Theorem 2.56 *Let ξ,η,τ be uncertain variables, and let $d(\cdot,\cdot)$ be the distance. Then we have*
(a) (Nonnegativity) $d(\xi,\eta) \ge 0$;
(b) (Identification) $d(\xi,\eta) = 0$ if and only if $\xi = \eta$;
(c) (Symmetry) $d(\xi,\eta) = d(\eta,\xi)$;
(d) (Triangle Inequality) $d(\xi,\eta) \le 2d(\xi,\tau) + 2d(\eta,\tau)$.

Proof: The parts (a), (b) and (c) follow immediately from the definition. Now we prove the part (d). It follows from the subadditivity axiom that

$$\begin{aligned}
d(\xi,\eta) &= \int_0^{+\infty} \mathcal{M}\left\{|\xi-\eta| \ge x\right\} \mathrm{d}x \\
&\le \int_0^{+\infty} \mathcal{M}\left\{|\xi-\tau| + |\tau-\eta| \ge x\right\} \mathrm{d}x \\
&\le \int_0^{+\infty} \mathcal{M}\left\{(|\xi-\tau| \ge x/2) \cup (|\tau-\eta| \ge x/2)\right\} \mathrm{d}x \\
&\le \int_0^{+\infty} \left(\mathcal{M}\{|\xi-\tau| \ge x/2\} + \mathcal{M}\{|\tau-\eta| \ge x/2\}\right) \mathrm{d}x \\
&= 2E[|\xi-\tau|] + 2E[|\tau-\eta|] = 2d(\xi,\tau) + 2d(\tau,\eta).
\end{aligned}$$

Example 2.17: Let $\Gamma = \{\gamma_1,\gamma_2,\gamma_3\}$. Define $\mathcal{M}\{\emptyset\} = 0$, $\mathcal{M}\{\Gamma\} = 1$ and $\mathcal{M}\{\Lambda\} = 1/2$ for any subset Λ (excluding $\emptyset$ and Γ). We set uncertain variables ξ, η and τ as follows,

$$\xi(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2 \\ 0, & \text{if } \gamma = \gamma_3, \end{cases} \qquad \eta(\gamma) = \begin{cases} 0, & \text{if } \gamma = \gamma_1 \\ -1, & \text{if } \gamma = \gamma_2 \\ -1, & \text{if } \gamma = \gamma_3, \end{cases} \qquad \tau(\gamma) \equiv 0.$$

It is easy to verify that $d(\xi,\tau) = d(\tau,\eta) = 1/2$ and $d(\xi,\eta) = 3/2$. Thus

$$d(\xi,\eta) = \frac{3}{2}(d(\xi,\tau) + d(\tau,\eta)).$$

A conjecture is $d(\xi,\eta) \le 1.5(d(\xi,\tau)+d(\tau,\eta))$ for arbitrary uncertain variables ξ, η and τ. This is an open problem.

How to Obtain Distance from Uncertainty Distributions?

Let ξ and η be independent uncertain variables. If $\xi-\eta$ has an uncertainty distribution Υ, then the distance is

$$\begin{aligned} d(\xi,\eta) &= \int_0^{+\infty} \mathcal{M}\{|\xi-\eta|\ge x\}\mathrm{d}x \\ &= \int_0^{+\infty} \mathcal{M}\{(\xi-\eta\ge x)\cup(\xi-\eta\le -x)\}\mathrm{d}x \\ &\le \int_0^{+\infty} (\mathcal{M}\{\xi-\eta\ge x\}+\mathcal{M}\{\xi-\eta\le -x\})\mathrm{d}x \\ &= \int_0^{+\infty} (1-\Upsilon(x)+\Upsilon(-x))\mathrm{d}x. \end{aligned}$$

Thus we have the following stipulation.

Stipulation 2.5 *Let ξ and η be independent uncertain variables, and let Υ be the uncertainty distribution of $\xi-\eta$. Then the distance between ξ and η is*

$$d(\xi,\eta) = \int_0^{+\infty} (1-\Upsilon(x)+\Upsilon(-x))\mathrm{d}x. \tag{2.178}$$

Theorem 2.57 *Let ξ and η be independent uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Then the distance between ξ and η is*

$$d(\xi,\eta) = \int_0^1 |\Phi^{-1}(\alpha)-\Psi^{-1}(1-\alpha)|\mathrm{d}\alpha. \tag{2.179}$$

Proof: Assume $\xi-\eta$ has an uncertainty distribution Υ. Substituting $\Upsilon(x)$ with α and x with $\Upsilon^{-1}(\alpha)$, the change of variables and integration by parts produce

$$\int_0^{+\infty}(1-\Upsilon(x))\mathrm{d}x = \int_{\Upsilon(0)}^1 (1-\alpha)\mathrm{d}\Upsilon^{-1}(\alpha) = \int_{\Upsilon(0)}^1 \Upsilon^{-1}(\alpha)\mathrm{d}\alpha.$$

Similarly, substituting $\Upsilon(-x)$ with α and x with $-\Upsilon^{-1}(\alpha)$, we obtain

$$\int_0^{+\infty}\Upsilon(-x)\mathrm{d}x = \int_{\Upsilon(0)}^0 \alpha\mathrm{d}(-\Upsilon^{-1}(\alpha)) = -\int_0^{\Upsilon(0)} \Upsilon^{-1}(\alpha)\mathrm{d}\alpha.$$

Based on the stipulation (2.178), we have

$$d(\xi,\eta) = \int_{\Upsilon(0)}^1 \Upsilon^{-1}(\alpha)\mathrm{d}\alpha - \int_0^{\Upsilon(0)} \Upsilon^{-1}(\alpha)\mathrm{d}\alpha = \int_0^1 |\Upsilon^{-1}(\alpha)|\mathrm{d}\alpha.$$

Since $\Upsilon^{-1}(\alpha)=\Phi^{-1}(\alpha)-\Psi^{-1}(1-\alpha)$, we immediately obtain the result.

2.10 Conditional Uncertainty Distribution

Definition 2.20 *(Liu [122]) The conditional uncertainty distribution Φ of an uncertain variable ξ given B is defined by*

$$\Phi(x|B) = \mathcal{M}\{\xi \le x|B\} \tag{2.180}$$

provided that $\mathcal{M}\{B\} > 0$.

Theorem 2.58 *(Liu [129]) Let ξ be an uncertain variable with uncertainty distribution $\Phi(x)$, and let t be a real number with $\Phi(t) < 1$. Then the conditional uncertainty distribution of ξ given $\xi > t$ is*

$$\Phi(x|(t,+\infty)) = \begin{cases} 0, & \text{if } \Phi(x) \le \Phi(t) \\ \dfrac{\Phi(x)}{1-\Phi(t)} \wedge 0.5, & \text{if } \Phi(t) < \Phi(x) \le (1+\Phi(t))/2 \\ \dfrac{\Phi(x)-\Phi(t)}{1-\Phi(t)}, & \text{if } (1+\Phi(t))/2 \le \Phi(x). \end{cases}$$

Proof: It follows from $\Phi(x|(t,+\infty)) = \mathcal{M}\{\xi \le x|\xi > t\}$ and the definition of conditional uncertainty that

$$\Phi(x|(t,+\infty)) = \begin{cases} \dfrac{\mathcal{M}\{(\xi \le x)\cap(\xi > t)\}}{\mathcal{M}\{\xi > t\}}, & \text{if } \dfrac{\mathcal{M}\{(\xi \le x)\cap(\xi > t)\}}{\mathcal{M}\{\xi > t\}} < 0.5 \\ 1 - \dfrac{\mathcal{M}\{(\xi > x)\cap(\xi > t)\}}{\mathcal{M}\{\xi > t\}}, & \text{if } \dfrac{\mathcal{M}\{(\xi > x)\cap(\xi > t)\}}{\mathcal{M}\{\xi > t\}} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

When $\Phi(x) \le \Phi(t)$, we have $x \le t$, and

$$\frac{\mathcal{M}\{(\xi \le x)\cap(\xi > t)\}}{\mathcal{M}\{\xi > t\}} = \frac{\mathcal{M}\{\emptyset\}}{1-\Phi(t)} = 0 < 0.5.$$

Thus

$$\Phi(x|(t,+\infty)) = \frac{\mathcal{M}\{(\xi \le x)\cap(\xi > t)\}}{\mathcal{M}\{\xi > t\}} = 0.$$

When $\Phi(t) < \Phi(x) \le (1+\Phi(t))/2$, we have $x > t$, and

$$\frac{\mathcal{M}\{(\xi > x)\cap(\xi > t)\}}{\mathcal{M}\{\xi > t\}} = \frac{1-\Phi(x)}{1-\Phi(t)} \ge \frac{1-(1+\Phi(t))/2}{1-\Phi(t)} = 0.5$$

and

$$\frac{\mathcal{M}\{(\xi \le x)\cap(\xi > t)\}}{\mathcal{M}\{\xi > t\}} \le \frac{\Phi(x)}{1-\Phi(t)}.$$

It follows from the maximum uncertainty principle that

$$\Phi(x|(t,+\infty)) = \frac{\Phi(x)}{1-\Phi(t)} \wedge 0.5.$$

When $(1+\Phi(t))/2 \le \Phi(x)$, we have $x \ge t$, and

$$\frac{\mathcal{M}\{(\xi > x)\cap(\xi > t)\}}{\mathcal{M}\{\xi > t\}} = \frac{1-\Phi(x)}{1-\Phi(t)} \le \frac{1-(1+\Phi(t))/2}{1-\Phi(t)} \le 0.5.$$

Thus

$$\Phi(x|(t,+\infty)) = 1 - \frac{\mathcal{M}\{(\xi > x)\cap(\xi > t)\}}{\mathcal{M}\{\xi > t\}} = 1 - \frac{1-\Phi(x)}{1-\Phi(t)} = \frac{\Phi(x)-\Phi(t)}{1-\Phi(t)}.$$

The theorem is proved.

Exercise 2.49: Let ξ be a linear uncertain variable $\mathcal{L}(a,b)$, and let t be a real number with $a < t < b$. Show that the conditional uncertainty distribution of ξ given $\xi > t$ is

$$\Phi(x|(t,+\infty)) = \begin{cases} 0, & \text{if } x \le t \\ \dfrac{x-a}{b-t} \wedge 0.5, & \text{if } t < x \le (b+t)/2 \\ \dfrac{x-t}{b-t} \wedge 1, & \text{if } (b+t)/2 \le x. \end{cases}$$

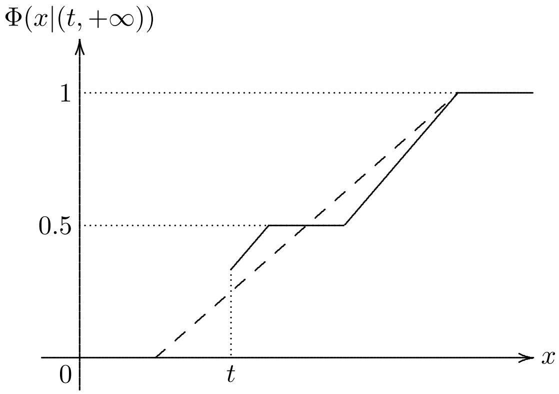

Figure 2.19: Conditional Uncertainty Distribution $\Phi(x|(t,+\infty))$

Theorem 2.59 *(Liu [129]) Let ξ be an uncertain variable with uncertainty distribution $\Phi(x)$, and let t be a real number with $\Phi(t) > 0$. Then the conditional uncertainty distribution of ξ given $\xi \le t$ is*

$$\Phi(x|(-\infty,t]) = \begin{cases} \dfrac{\Phi(x)}{\Phi(t)}, & \textit{if } \Phi(x) \le \Phi(t)/2 \\ \dfrac{\Phi(x)+\Phi(t)-1}{\Phi(t)} \vee 0.5, & \textit{if } \Phi(t)/2 \le \Phi(x) < \Phi(t) \\ 1, & \textit{if } \Phi(t) \le \Phi(x). \end{cases}$$

Proof: It follows from $\Phi(x|(-\infty,t]) = \mathcal{M}\{\xi \le x|\xi \le t\}$ and the definition of conditional uncertainty that

$$\Phi(x|(-\infty,t])=\begin{cases} \dfrac{\mathcal{M}\{(\xi\le x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}}, & \text{if } \dfrac{\mathcal{M}\{(\xi\le x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}}<0.5\\ 1-\dfrac{\mathcal{M}\{(\xi> x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}}, & \text{if } \dfrac{\mathcal{M}\{(\xi> x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}}<0.5\\ 0.5, & \text{otherwise.}\end{cases}$$

When $\Phi(x) \le \Phi(t)/2$, we have $x < t$, and

$$\frac{\mathcal{M}\{(\xi\le x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}} = \frac{\Phi(x)}{\Phi(t)} \le \frac{\Phi(t)/2}{\Phi(t)} = 0.5.$$

Thus

$$\Phi(x|(-\infty,t]) = \frac{\mathcal{M}\{(\xi\le x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}} = \frac{\Phi(x)}{\Phi(t)}.$$

When $\Phi(t)/2 \le \Phi(x) < \Phi(t)$, we have $x < t$, and

$$\frac{\mathcal{M}\{(\xi\le x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}} = \frac{\Phi(x)}{\Phi(t)} \ge \frac{\Phi(t)/2}{\Phi(t)} = 0.5$$

and

$$\frac{\mathcal{M}\{(\xi> x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}} \le \frac{1-\Phi(x)}{\Phi(t)},$$

i.e.,

$$1-\frac{\mathcal{M}\{(\xi> x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}} \ge \frac{\Phi(x)+\Phi(t)-1}{\Phi(t)}.$$

It follows from the maximum uncertainty principle that

$$\Phi(x|(-\infty,t]) = \frac{\Phi(x)+\Phi(t)-1}{\Phi(t)} \vee 0.5.$$

When $\Phi(t) \le \Phi(x)$, we have $x \ge t$, and

$$\frac{\mathcal{M}\{(\xi> x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}} = \frac{\mathcal{M}\{\emptyset\}}{\Phi(t)} = 0 < 0.5.$$

Thus

$$\Phi(x|(-\infty,t]) = 1-\frac{\mathcal{M}\{(\xi> x)\cap(\xi\le t)\}}{\mathcal{M}\{\xi\le t\}} = 1-0=1.$$

The theorem is proved.

Exercise 2.50: Let ξ be a linear uncertain variable $\mathcal{L}(a,b)$, and let t be a real number with $a < t < b$. Show that the conditional uncertainty distribution of ξ given $\xi \le t$ is

$$\Phi(x|(-\infty,t]) = \begin{cases} \dfrac{x-a}{t-a} \vee 0, & \text{if } x \le (a+t)/2 \\ \left(1 - \dfrac{b-x}{t-a}\right) \vee 0.5, & \text{if } (a+t)/2 \le x < t \\ 1, & \text{if } x \ge t. \end{cases}$$

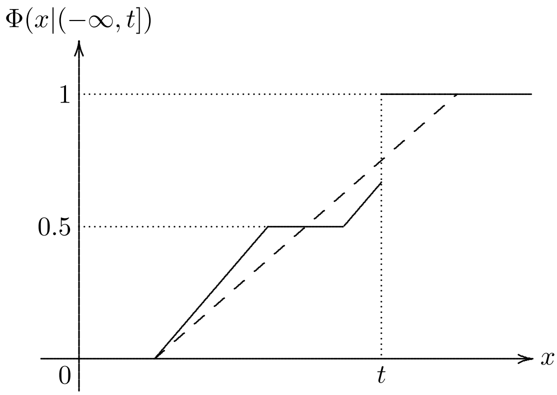

Figure 2.20: Conditional Uncertainty Distribution $\Phi(x|(-\infty,t])$

2.11 Uncertain Sequence

Uncertain sequence is a sequence of uncertain variables indexed by integers. This section introduces four convergence concepts of uncertain sequence: convergence almost surely (a.s.), convergence in measure, convergence in mean, and convergence in distribution.

Table 2.1: Relationship among Convergence Concepts

Convergence in Mean	$\Rightarrow$	Convergence in Measure	$\Rightarrow$	Convergence in Distribution
		Convergence Almost Surely		

Definition 2.21 *(Liu [122]) The uncertain sequence $\{\xi_i\}$ is said to be convergent a.s. to ξ if there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$$\lim_{i\to\infty} |\xi_i(\gamma) - \xi(\gamma)| = 0 \tag{2.181}$$

for every $\gamma \in \Lambda$. In that case we write $\xi_i \to \xi$, a.s.

Definition 2.22 *(Liu [122]) The uncertain sequence $\{\xi_i\}$ is said to be convergent in measure to ξ if*

$$\lim_{i\to\infty} \mathcal{M}\{|\xi_i - \xi| \ge \varepsilon\} = 0 \tag{2.182}$$

for every $\varepsilon > 0$.

Definition 2.23 *(Liu [122]) The uncertain sequence $\{\xi_i\}$ is said to be convergent in mean to ξ if*

$$\lim_{i\to\infty} E[|\xi_i - \xi|] = 0. \tag{2.183}$$

Definition 2.24 *(Liu [122]) Let $\Phi, \Phi_1, \Phi_2, \cdots$ be the uncertainty distributions of uncertain variables $\xi, \xi_1, \xi_2, \cdots$, respectively. We say the uncertain sequence $\{\xi_i\}$ converges in distribution to ξ if*

$$\lim_{i\to\infty} \Phi_i(x) = \Phi(x) \tag{2.184}$$

for all x at which $\Phi(x)$ is continuous.

Convergence in Mean vs. Convergence in Measure

Theorem 2.60 *(Liu [122]) If the uncertain sequence $\{\xi_i\}$ converges in mean to ξ, then $\{\xi_i\}$ converges in measure to ξ.*

Proof: It follows from the Markov inequality that for any given number $\varepsilon > 0$, we have

$$\mathcal{M}\{|\xi_i - \xi| \ge \varepsilon\} \le \frac{E[|\xi_i - \xi|]}{\varepsilon} \to 0$$

as $i \to \infty$. Thus $\{\xi_i\}$ converges in measure to ξ. The theorem is proved.

Example 2.18: Convergence in measure does not imply convergence in mean. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_i \in \Lambda} 1/i, & \text{if } \sup\limits_{\gamma_i \in \Lambda} 1/i < 0.5 \\ 1 - \sup\limits_{\gamma_i \notin \Lambda} 1/i, & \text{if } \sup\limits_{\gamma_i \notin \Lambda} 1/i < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

The uncertain variables are defined by

$$\xi_i(\gamma_j) = \begin{cases} i, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. For some small number $\varepsilon > 0$, we have

$$\mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = \mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = \frac{1}{i} \to 0$$

as $i \to \infty$. That is, the sequence $\{\xi_i\}$ converges in measure to ξ. However, for each i, we have

$$E[|\xi_i - \xi|] = 1.$$

That is, the sequence $\{\xi_i\}$ does not converge in mean to ξ.

Convergence in Measure vs. Convergence in Distribution

Theorem 2.61 *(Liu [122]) If the uncertain sequence $\{\xi_i\}$ converges in measure to ξ, then $\{\xi_i\}$ converges in distribution to ξ.*

Proof: Let x be a given continuity point of the uncertainty distribution Φ. On the one hand, for any $y > x$, we have

$$\{\xi_i \leq x\} = \{\xi_i \leq x, \xi \leq y\} \cup \{\xi_i \leq x, \xi > y\} \subset \{\xi \leq y\} \cup \{|\xi_i - \xi| \geq y - x\}.$$

It follows from the subadditivity axiom that

$$\Phi_i(x) \leq \Phi(y) + \mathcal{M}\{|\xi_i - \xi| \geq y - x\}.$$

Since $\{\xi_i\}$ converges in measure to ξ, we have $\mathcal{M}\{|\xi_i - \xi| \geq y - x\} \to 0$ as $i \to \infty$. Thus we obtain $\limsup_{i\to\infty} \Phi_i(x) \leq \Phi(y)$ for any $y > x$. Letting $y \to x$, we get

$$\limsup_{i\to\infty} \Phi_i(x) \leq \Phi(x). \tag{2.185}$$

On the other hand, for any $z < x$, we have

$$\{\xi \leq z\} = \{\xi_i \leq x, \xi \leq z\} \cup \{\xi_i > x, \xi \leq z\} \subset \{\xi_i \leq x\} \cup \{|\xi_i - \xi| \geq x - z\}$$

which implies that

$$\Phi(z) \leq \Phi_i(x) + \mathcal{M}\{|\xi_i - \xi| \geq x - z\}.$$

Since $\mathcal{M}\{|\xi_i - \xi| \geq x - z\} \to 0$, we obtain $\Phi(z) \leq \liminf_{i\to\infty} \Phi_i(x)$ for any $z < x$. Letting $z \to x$, we get

$$\Phi(x) \leq \liminf_{i\to\infty} \Phi_i(x). \tag{2.186}$$

It follows from (2.185) and (2.186) that $\Phi_i(x) \to \Phi(x)$ as $i \to \infty$. The theorem is proved.

Example 2.19: Convergence in distribution does not imply convergence in measure. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 1/2$. We define an uncertain variable as

$$\xi(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2. \end{cases}$$

We also define $\xi_i = -\xi$ for $i = 1, 2, \cdots$ Then ξ_i and ξ have the same chance distribution. Thus $\{\xi_i\}$ converges in distribution to ξ. However, for some small number $\varepsilon > 0$, we have

$$\mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = \mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = 1.$$

That is, the sequence $\{\xi_i\}$ does not converge in measure to ξ.

Convergence Almost Surely vs. Convergence in Measure

Example 2.20: Convergence a.s. does not imply convergence in measure. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_i \in \Lambda} i/(2i+1), & \text{if } \sup\limits_{\gamma_i \in \Lambda} i/(2i+1) < 0.5 \\ 1 - \sup\limits_{\gamma_i \notin \Lambda} i/(2i+1), & \text{if } \sup\limits_{\gamma_i \notin \Lambda} i/(2i+1) < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

Then we define uncertain variables as

$$\xi_i(\gamma_j) = \begin{cases} i, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. The sequence $\{\xi_i\}$ converges a.s. to ξ. However, for some small number $\varepsilon > 0$, we have

$$\mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = \mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = \frac{i}{2i+1} \to \frac{1}{2}$$

as $i \to \infty$. That is, the sequence $\{\xi_i\}$ does not converge in measure to ξ.

Example 2.21: Convergence in measure does not imply convergence a.s. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. For any positive integer i, there is an integer j such that $i = 2^j + k$, where k is an integer between 0 and $2^j - 1$. Then we define uncertain variables as

$$\xi_i(\gamma) = \begin{cases} 1, & \text{if } k/2^j \leq \gamma \leq (k+1)/2^j \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. For some small number $\varepsilon > 0$, we have

$$\mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = \mathcal{M}\{|\xi_i - \xi| \geq \varepsilon\} = \frac{1}{2^j} \to 0$$

as $i \to \infty$. That is, the sequence $\{\xi_i\}$ converges in measure to ξ. However, for any $\gamma \in [0, 1]$, there is an infinite number of intervals of the form $[k/2^j, (k+1)/2^j]$ containing γ. Thus $\xi_i(\gamma)$ does not converge to 0. In other words, the sequence $\{\xi_i\}$ does not converge a.s. to ξ.

Convergence Almost Surely vs. Convergence in Mean

Example 2.22: Convergence a.s. does not imply convergence in mean. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with

$$\mathcal{M}\{\Lambda\} = \sum_{\gamma_i \in \Lambda} \frac{1}{2^i}.$$

The uncertain variables are defined by

$$\xi_i(\gamma_j) = \begin{cases} 2^i, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. Then ξ_i converges a.s. to ξ. However, the sequence $\{\xi_i\}$ does not converge in mean to ξ because $E[|\xi_i - \xi|] \equiv 1$ for each i.

Example 2.23: Convergence in mean does not imply convergence a.s. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. For any positive integer i, there is an integer j such that $i = 2^j + k$, where k is an integer between 0 and $2^j - 1$. The uncertain variables are defined by

$$\xi_i(\gamma) = \begin{cases} 1, & \text{if } k/2^j \le \gamma \le (k+1)/2^j \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. Then

$$E[|\xi_i - \xi|] = \frac{1}{2^j} \to 0$$

as $i \to \infty$. That is, the sequence $\{\xi_i\}$ converges in mean to ξ. However, for any $\gamma \in [0, 1]$, there is an infinite number of intervals of the form $[k/2^j, (k+1)/2^j]$ containing γ. Thus $\xi_i(\gamma)$ does not converge to 0. In other words, the sequence $\{\xi_i\}$ does not converge a.s. to ξ.

Convergence Almost Surely vs. Convergence in Distribution

Example 2.24: Convergence in distribution does not imply convergence a.s. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with $\mathcal{M}\{\gamma_1\} = \mathcal{M}\{\gamma_2\} = 1/2$. We define an uncertain variable ξ as

$$\xi(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 1, & \text{if } \gamma = \gamma_2. \end{cases}$$

We also define $\xi_i = -\xi$ for $i = 1, 2, \cdots$ Then ξ_i and ξ have the same uncertainty distribution. Thus $\{\xi_i\}$ converges in distribution to ξ. However, the sequence $\{\xi_i\}$ does not converge a.s. to ξ.

Example 2.25: Convergence a.s. does not imply convergence in distribution. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \cdots\}$ with

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{\gamma_i \in \Lambda} i/(2i+1), & \text{if } \sup\limits_{\gamma_i \in \Lambda} i/(2i+1) < 0.5 \\ 1 - \sup\limits_{\gamma_i \notin \Lambda} i/(2i+1), & \text{if } \sup\limits_{\gamma_i \notin \Lambda} i/(2i+1) < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

The uncertain variables are defined by

$$\xi_i(\gamma_j) = \begin{cases} i, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi \equiv 0$. Then the sequence $\{\xi_i\}$ converges a.s. to ξ. However, the uncertainty distributions of ξ_i are

$$\Phi_i(x) = \begin{cases} 0, & \text{if } x < 0 \\ (i+1)/(2i+1), & \text{if } 0 \le x < i \\ 1, & \text{if } x \ge i \end{cases}$$

for $i = 1, 2, \cdots$, respectively. The uncertainty distribution of ξ is

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \ge 0. \end{cases}$$

It is clear that $\Phi_i(x)$ does not converge to $\Phi(x)$ at $x > 0$. That is, the sequence $\{\xi_i\}$ does not converge in distribution to ξ.

2.12 Uncertain Vector

As an extension of uncertain variable, this section introduces a concept of uncertain vector whose components are uncertain variables.

Definition 2.25 *(Liu [122]) A k-dimensional uncertain vector is a function $\boldsymbol{\xi}$ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of k-dimensional real vectors such that $\{\boldsymbol{\xi} \in B\}$ is an event for any k-dimensional Borel set B.*

Theorem 2.62 *(Liu [122]) The vector $(\xi_1, \xi_2, \cdots, \xi_k)$ is an uncertain vector if and only if $\xi_1, \xi_2, \cdots, \xi_k$ are uncertain variables.*

Proof: Write $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_k)$. Suppose that $\boldsymbol{\xi}$ is an uncertain vector on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. For any Borel set B over $\Re$, the set $B \times \Re^{k-1}$ is a k-dimensional Borel set. Thus the set

$$\{\xi_1 \in B\} = \{\xi_1 \in B, \xi_2 \in \Re, \cdots, \xi_k \in \Re\} = \{\boldsymbol{\xi} \in B \times \Re^{k-1}\}$$

is an event. Hence ξ_1 is an uncertain variable. A similar process may prove that $\xi_2, \xi_3, \cdots, \xi_k$ are uncertain variables.

Conversely, suppose that all $\xi_1, \xi_2, \cdots, \xi_k$ are uncertain variables on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. We define

$$\mathcal{B} = \left\{B \subset \Re^k \;\middle|\; \{\boldsymbol{\xi} \in B\} \text{ is an event}\right\}.$$

The vector $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_k)$ is proved to be an uncertain vector if we can prove that $\mathcal{B}$ contains all k-dimensional Borel sets. First, the class $\mathcal{B}$ contains all open intervals of $\Re^k$ because

$$\left\{\boldsymbol{\xi} \in \prod_{i=1}^{k}(a_i, b_i)\right\} = \bigcap_{i=1}^{k} \{\xi_i \in (a_i, b_i)\}$$

is an event. Next, the class $\mathcal{B}$ is a σ-algebra over $\Re^k$ because (i) we have $\Re^k \in \mathcal{B}$ since $\{\boldsymbol{\xi} \in \Re^k\} = \Gamma$; (ii) if $B \in \mathcal{B}$, then $\{\boldsymbol{\xi} \in B\}$ is an event, and

$$\{\boldsymbol{\xi} \in B^c\} = \{\boldsymbol{\xi} \in B\}^c$$

is an event. This means that $B^c \in \mathcal{B}$; (iii) if $B_i \in \mathcal{B}$ for $i = 1, 2, \cdots$, then $\{\boldsymbol{\xi} \in B_i\}$ are events and

$$\left\{\boldsymbol{\xi} \in \bigcup_{i=1}^{\infty} B_i\right\} = \bigcup_{i=1}^{\infty}\{\boldsymbol{\xi} \in B_i\}$$

is an event. This means that $\cup_i B_i \in \mathcal{B}$. Since the smallest σ-algebra containing all open intervals of $\Re^k$ is just the Borel algebra over $\Re^k$, the class $\mathcal{B}$ contains all k-dimensional Borel sets. The theorem is proved.

Definition 2.26 *(Liu [122]) The joint uncertainty distribution of an uncertain vector $(\xi_1, \xi_2, \cdots, \xi_k)$ is defined by*

$$\Phi(x_1, x_2, \cdots, x_k) = \mathcal{M}\{\xi_1 \le x_1, \xi_2 \le x_2, \cdots, \xi_k \le x_k\} \tag{2.187}$$

for any real numbers $x_1, x_2, \cdots, x_k$.

Theorem 2.63 *(Liu [122]) Let $\xi_1, \xi_2, \cdots, \xi_k$ be independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_k$, respectively. Then the uncertain vector $(\xi_1, \xi_2, \cdots, \xi_k)$ has a joint uncertainty distribution*

$$\Phi(x_1, x_2, \cdots, x_k) = \Phi_1(x_1) \wedge \Phi_2(x_2) \wedge \cdots \wedge \Phi_k(x_k) \tag{2.188}$$

for any real numbers $x_1, x_2, \cdots, x_k$.

Proof: Since $\xi_1, \xi_2, \cdots, \xi_k$ are independent uncertain variables, we have

$$\Phi(x_1, x_2, \cdots, x_k) = \mathcal{M}\left\{\bigcap_{i=1}^{k}(\xi_i \le x_i)\right\} = \bigwedge_{i=1}^{k} \mathcal{M}\{\xi_i \le x_i\} = \bigwedge_{i=1}^{k} \Phi_i(x_i)$$

for any real numbers $x_1, x_2, \cdots, x_k$. The theorem is proved.

Remark 2.9: However, the equation (2.188) does not imply that the uncertain variables are independent. For example, let ξ be an uncertain variable with uncertainty distribution Φ. Then the joint uncertainty distribution Ψ of uncertain vector (ξ, ξ) is

$$\Psi(x_1, x_2) = \mathcal{M}\{(\xi \le x_1) \cap (\xi \le x_2)\} = \Phi(x_1) \wedge \Phi(x_2)$$

for any real numbers x_1 and x_2. But, generally speaking, an uncertain variable is not independent with itself.

Definition 2.27 *(Liu [137]) The k-dimensional uncertain vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_n$ are said to be independent if for any k-dimensional Borel sets $B_1, B_2, \cdots, B_n$, we have*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in B_i\}. \tag{2.189}$$

Exercise 2.51: Let $(\xi_1, \xi_2, \cdots, \xi_k)$ and $(\eta_1, \eta_2, \cdots, \eta_k)$ be independent uncertain vectors. Show that ξ_1 and (η_1, η_k) are independent.

Theorem 2.64 *(Liu [137]) The k-dimensional uncertain vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_n$ are independent if and only if*

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in B_i\} \tag{2.190}$$

for any k-dimensional Borel sets $B_1, B_2, \cdots, B_n$.

Proof: It follows from the duality of uncertain measure that $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_n$ are independent if and only if

$$\begin{aligned}\mathcal{M}\left\{\bigcup_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i)\right\} &= 1 - \mathcal{M}\left\{\bigcap_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i^c)\right\} \\ &= 1 - \bigwedge_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in B_i^c\} = \bigvee_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in B_i\}.\end{aligned}$$

The theorem is thus proved.

Theorem 2.65 *Let $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_n$ be independent uncertain vectors, and let $f_1, f_2, \cdots, f_n$ be vector-valued measurable functions. Then $f_1(\boldsymbol{\xi}_1), f_2(\boldsymbol{\xi}_2), \cdots, f_n(\boldsymbol{\xi}_n)$ are also independent uncertain vectors.*

Proof: For any Borel sets $B_1, B_2, \cdots, B_n$, it follows from the definition of independence that

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(f_i(\boldsymbol{\xi}_i) \in B_i)\right\} = \mathcal{M}\left\{\bigcap_{i=1}^{n}(\boldsymbol{\xi}_i \in f_i^{-1}(B_i))\right\}$$

$$= \bigwedge_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in f_i^{-1}(B_i)\} = \bigwedge_{i=1}^{n} \mathcal{M}\{f_i(\boldsymbol{\xi}_i) \in B_i\}.$$

Thus $f_1(\boldsymbol{\xi}_1), f_2(\boldsymbol{\xi}_2), \cdots, f_n(\boldsymbol{\xi}_n)$ are independent uncertain variables.

Normal Uncertain Vector

Definition 2.28 *(Liu [137]) Let $\tau_1, \tau_2, \cdots, \tau_m$ be independent normal uncertain variables with expected value* 0 *and variance* 1. *Then*

$$\boldsymbol{\tau} = (\tau_1, \tau_2, \cdots, \tau_m) \tag{2.191}$$

is called a standard normal uncertain vector.

It is easy to verify that a standard normal uncertain vector $(\tau_1, \tau_2, \cdots, \tau_m)$ has a joint uncertainty distribution

$$\Phi(x_1, x_2, \cdots, x_m) = \left(1 + \exp\left(-\frac{\pi(x_1 \wedge x_2 \wedge \cdots \wedge x_m)}{\sqrt{3}}\right)\right)^{-1} \tag{2.192}$$

for any real numbers $x_1, x_2, \cdots, x_m$. It is also easy to show that

$$\lim_{x_i \to -\infty} \Phi(x_1, x_2, \cdots, x_m) = 0, \quad \text{for each } i, \tag{2.193}$$

$$\lim_{(x_1, x_2, \cdots, x_m) \to +\infty} \Phi(x_1, x_2, \cdots, x_m) = 1. \tag{2.194}$$

Furthermore, the limit

$$\lim_{(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_m) \to +\infty} \Phi(x_1, x_2, \cdots, x_m) \tag{2.195}$$

is a standard normal distribution with respect to x_i.

Definition 2.29 *(Liu [137]) Let $(\tau_1, \tau_2, \cdots, \tau_m)$ be a standard normal uncertain vector, and let e_i, σ_{ij}, $i = 1, 2, \cdots, k$, $j = 1, 2, \cdots, m$ be real numbers. Define*

$$\xi_i = e_i + \sum_{j=1}^{m} \sigma_{ij} \tau_j \tag{2.196}$$

for $i = 1, 2, \cdots, k$. Then $(\xi_1, \xi_2, \cdots, \xi_k)$ is called a normal uncertain vector.

That is, an uncertain vector $\boldsymbol{\xi}$ has a multivariate normal distribution if it can be represented in the form

$$\boldsymbol{\xi} = \boldsymbol{e} + \boldsymbol{\sigma}\boldsymbol{\tau} \tag{2.197}$$

for some real vector $\boldsymbol{e}$ and some real matrix $\boldsymbol{\sigma}$, where $\boldsymbol{\tau}$ is a standard normal uncertain vector. Note that $\boldsymbol{\xi}, \boldsymbol{e}$ and $\boldsymbol{\tau}$ are understood as column vectors. Please also note that for every index i, the component ξ_i is a normal uncertain variable with expected value e_i and standard variance

$$\sum_{j=1}^{m} |\sigma_{ij}|. \tag{2.198}$$

Theorem 2.66 *(Liu [137]) Assume $\boldsymbol{\xi}$ is a normal uncertain vector, $\boldsymbol{c}$ is a real vector, and D is a real matrix. Then*

$$\boldsymbol{\eta} = \boldsymbol{c} + D\boldsymbol{\xi} \tag{2.199}$$

is another normal uncertain vector.

Proof: Since $\boldsymbol{\xi}$ is a normal uncertain vector, there exists a standard normal uncertain vector $\boldsymbol{\tau}$, a real vector $\boldsymbol{e}$ and a real matrix $\boldsymbol{\sigma}$ such that $\boldsymbol{\xi} = \boldsymbol{e} + \boldsymbol{\sigma}\boldsymbol{\tau}$. It follows that

$$\boldsymbol{\eta} = \boldsymbol{c} + D\boldsymbol{\xi} = \boldsymbol{c} + D(\boldsymbol{e} + \boldsymbol{\sigma}\boldsymbol{\tau}) = (\boldsymbol{c} + D\boldsymbol{e}) + (D\boldsymbol{\sigma})\boldsymbol{\tau}.$$

Hence $\boldsymbol{\eta}$ is a normal uncertain vector.

2.13 Bibliographic Notes

As a fundamental concept in uncertainty theory, the uncertain variable was presented by Liu [122] in 2007. In order to describe uncertain variable, Liu [122] also introduced the concept of uncertainty distribution. Later, Peng and Iwamura [184] proved a sufficient and necessary condition for uncertainty distribution. In addition, Liu [129] proposed the concept of inverse uncertainty distribution, and Liu [134] verified a sufficient and necessary condition for it. More importantly, a measure inversion theorem was given by Liu [129] that may yield uncertain measures from the uncertainty distribution of the corresponding uncertain variable. Furthermore, Liu [122] proposed the concept of conditional uncertainty distribution of uncertain variable, and derived some formulas for calculating it.

Following the independence concept of uncertain variables proposed by Liu [125], the operational law was given by Liu [129] for calculating the uncertainty distribution and inverse uncertainty distribution of strictly monotone function of independent uncertain variables.

In order to rank uncertain variables, Liu [122] proposed the concept of expected value operator. In addition, the linearity of expected value operator was verified by Liu [129]. As an important contribution, Liu and Ha [147] derived a useful formula for calculating the expected values of strictly monotone functions of independent uncertain variables. Based on the expected value operator, Liu [122] presented the concepts of variance, moments and distance of uncertain variables.

The concept of entropy was proposed by Liu [125] for characterizing the uncertainty of uncertain variables. Dai and Chen [27] verified the positive linearity of entropy and derived some formulas for calculating the entropy of monotone function of uncertain variables. In addition, Chen and Dai [15] discussed the maximum entropy principle in order to select the uncertainty distribution that has maximum entropy and satisfies the prescribed constraints. Especially, normal uncertainty distribution is proved to have maximum entropy when the expected value and variance are fixed in advance. As an extension of entropy, Chen, Kar and Ralescu [16] proposed a concept of cross entropy for comparing an uncertainty distribution against a reference uncertainty distribution.

The concept of uncertain sequence was presented by Liu [122] with convergence almost surely, convergence in measure, convergence in mean, and convergence in distribution. Liu [122] also discussed the relationship among those convergence concepts. Furthermore, Gao [48], You [258], Zhang [268], and Chen, Li and Ralescu [22] developed some other concepts of convergence and investigated their mathematical properties.

The concept of uncertain vector was defined by Liu [122]. In addition, Liu [137] discussed the independence of uncertain vectors and proposed the concept of normal uncertain vector.

Chapter 3

Uncertain Programming

Uncertain programming was founded by Liu [124] in 2009. This chapter will provide a theory of uncertain programming, and present some uncertain programming models for machine scheduling problem, vehicle routing problem, and project scheduling problem.

3.1 Uncertain Programming

Uncertain programming is a type of mathematical programming involving uncertain variables. Assume that $\boldsymbol{x}$ is a decision vector, and $\boldsymbol{\xi}$ is an uncertain vector. Since an uncertain objective function $f(\boldsymbol{x}, \boldsymbol{\xi})$ cannot be directly minimized, we may minimize its expected value, i.e.,

$$\min_{\boldsymbol{x}} E[f(\boldsymbol{x}, \boldsymbol{\xi})]. \tag{3.1}$$

In addition, since the uncertain constraints $g_j(\boldsymbol{x}, \boldsymbol{\xi}) \le 0, j = 1, 2, \cdots, p$ do not define a crisp feasible set, it is naturally desired that the uncertain constraints hold with confidence levels $\alpha_1, \alpha_2, \cdots, \alpha_p$. Then we have a set of chance constraints,

$$\mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \le 0\} \ge \alpha_j, \quad j = 1, 2, \cdots, p. \tag{3.2}$$

In order to obtain a decision with minimum expected objective value subject to a set of chance constraints, Liu [124] proposed the following uncertain programming model,

$$\begin{cases} \min\limits_{\boldsymbol{x}} E[f(\boldsymbol{x}, \boldsymbol{\xi})] \\ \text{subject to:} \\ \qquad \mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \le 0\} \ge \alpha_j, \quad j = 1, 2, \cdots, p. \end{cases} \tag{3.3}$$

Definition 3.1 *(Liu [124]) A vector $\boldsymbol{x}$ is called a feasible solution to the uncertain programming model (3.3) if*

$$\mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \le 0\} \ge \alpha_j \tag{3.4}$$

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5_4

for $j = 1, 2, \cdots, p$.

Definition 3.2 *(Liu [124]) A feasible solution* $\boldsymbol{x}^*$ *is called an optimal solution to the uncertain programming model (3.3) if*

$$E[f(\boldsymbol{x}^*, \boldsymbol{\xi})] \le E[f(\boldsymbol{x}, \boldsymbol{\xi})] \tag{3.5}$$

for any feasible solution $\boldsymbol{x}$.

Theorem 3.1 *Assume the objective function* $f(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n)$ *is strictly increasing with respect to* $\xi_1, \xi_2, \cdots, \xi_m$ *and strictly decreasing with respect to* $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$. *If* $\xi_1, \xi_2, \cdots, \xi_n$ *are independent uncertain variables with uncertainty distributions* $\Phi_1, \Phi_2, \cdots, \Phi_n$, *respectively, then the expected objective function* $E[f(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n)]$ *is equal to*

$$\int_0^1 f(\boldsymbol{x}, \Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha. \tag{3.6}$$

Proof: It follows from Theorem 2.30 immediately.

Exercise 3.1: Assume $f(\boldsymbol{x}, \boldsymbol{\xi}) = h_1(\boldsymbol{x})\xi_1 + h_2(\boldsymbol{x})\xi_2 + \cdots + h_n(\boldsymbol{x})\xi_n + h_0(\boldsymbol{x})$ where $h_1(\boldsymbol{x}), h_2(\boldsymbol{x}), \cdots, h_n(\boldsymbol{x}), h_0(\boldsymbol{x})$ are real-valued functions and $\xi_1, \xi_2, \cdots,$ ξ_n are independent uncertain variables. Show that

$$E[f(\boldsymbol{x}, \boldsymbol{\xi})] = h_1(\boldsymbol{x})E[\xi_1] + h_2(\boldsymbol{x})E[\xi_2] + \cdots + h_n(\boldsymbol{x})E[\xi_n] + h_0(\boldsymbol{x}). \tag{3.7}$$

Theorem 3.2 *Assume the constraint function* $g(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n)$ *is strictly increasing with respect to* $\xi_1, \xi_2, \cdots, \xi_k$ *and strictly decreasing with respect to* $\xi_{k+1}, \xi_{k+2}, \cdots, \xi_n$. *If* $\xi_1, \xi_2, \cdots, \xi_n$ *are independent uncertain variables with uncertainty distributions* $\Phi_1, \Phi_2, \cdots, \Phi_n$, *respectively, then the chance constraint*

$$\mathcal{M}\{g(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n) \le 0\} \ge \alpha \tag{3.8}$$

holds if and only if

$$g(\boldsymbol{x}, \Phi_1^{-1}(\alpha), \cdots, \Phi_k^{-1}(\alpha), \Phi_{k+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)) \le 0. \tag{3.9}$$

Proof: It follows from Theorem 2.22 immediately.

Exercise 3.2: Assume $x_1, x_2, \cdots, x_n$ are nonnegative decision variables, and $\xi_1, \xi_2, \cdots, \xi_n, \xi$ are independent linear uncertain variables $\mathcal{L}(a_1, b_1), \mathcal{L}(a_2, b_2),$ $\cdots, \mathcal{L}(a_n, b_n), \mathcal{L}(a, b)$, respectively. Show that for any confidence level $\alpha \in$ $(0, 1)$, the chance constraint

$$\mathcal{M}\left\{\sum_{i=1}^n \xi_i x_i \le \xi\right\} \ge \alpha \tag{3.10}$$

holds if and only if

$$\sum_{i=1}^{n}((1-\alpha)a_i+\alpha b_i)x_i \le \alpha a+(1-\alpha)b. \tag{3.11}$$

Exercise 3.3: Assume $x_1, x_2, \cdots, x_n$ are nonnegative decision variables, and $\xi_1, \xi_2, \cdots, \xi_n, \xi$ are independent normal uncertain variables $\mathcal{N}(e_1, \sigma_1)$, $\mathcal{N}(e_2, \sigma_2), \cdots, \mathcal{N}(e_n, \sigma_n), \mathcal{N}(e, \sigma)$, respectively. Show that for any confidence level $\alpha \in (0,1)$, the chance constraint

$$\mathcal{M}\left\{\sum_{i=1}^{n}\xi_i x_i \le \xi\right\} \ge \alpha \tag{3.12}$$

holds if and only if

$$\sum_{i=1}^{n}\left(e_i+\frac{\sigma_i\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)x_i \le e-\frac{\sigma\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}. \tag{3.13}$$

Exercise 3.4: Assume $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, and $h_1(\boldsymbol{x})$, $h_2(\boldsymbol{x}), \cdots, h_n(\boldsymbol{x}), h_0(\boldsymbol{x})$ are real-valued functions. Show that

$$\mathcal{M}\left\{\sum_{i=1}^{n}h_i(\boldsymbol{x})\xi_i \le h_0(\boldsymbol{x})\right\} \ge \alpha \tag{3.14}$$

holds if and only if

$$\sum_{i=1}^{n}h_i^+(\boldsymbol{x})\Phi_i^{-1}(\alpha)-\sum_{i=1}^{n}h_i^-(\boldsymbol{x})\Phi_i^{-1}(1-\alpha) \le h_0(\boldsymbol{x}) \tag{3.15}$$

where

$$h_i^+(\boldsymbol{x})=\begin{cases} h_i(\boldsymbol{x}), & \text{if } h_i(\boldsymbol{x})>0\\ 0, & \text{if } h_i(\boldsymbol{x})\le 0,\end{cases} \tag{3.16}$$

$$h_i^-(\boldsymbol{x})=\begin{cases} -h_i(\boldsymbol{x}), & \text{if } h_i(\boldsymbol{x})<0\\ 0, & \text{if } h_i(\boldsymbol{x})\ge 0\end{cases} \tag{3.17}$$

for $i = 1, 2, \cdots, n$.

Theorem 3.3 *Assume $f(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, and $g_j(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n)$ are strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_k$ and strictly decreasing with respect to $\xi_{k+1}, \xi_{k+2}, \cdots, \xi_n$ for $j = 1, 2, \cdots, p$.*

If $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, then the uncertain programming

$$\begin{cases} \min\limits_{\boldsymbol{x}} E[f(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n)] \\ \textit{subject to:} \\ \qquad \mathcal{M}\{g_j(\boldsymbol{x}, \xi_1, \xi_2, \cdots, \xi_n) \le 0\} \ge \alpha_j, \quad j = 1, 2, \cdots, p \end{cases} \tag{3.18}$$

is equivalent to the crisp mathematical programming

$$\begin{cases} \min\limits_{\boldsymbol{x}} \displaystyle\int_0^1 f(\boldsymbol{x}, \Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha \\ \textit{subject to:} \\ \qquad g_j(\boldsymbol{x}, \Phi_1^{-1}(\alpha_j), \cdots, \Phi_k^{-1}(\alpha_j), \Phi_{k+1}^{-1}(1-\alpha_j), \cdots, \Phi_n^{-1}(1-\alpha_j)) \le 0 \\ \qquad j = 1, 2, \cdots, p. \end{cases}$$

Proof: It follows from Theorems 3.1 and 3.2 immediately.

3.2 Numerical Method

When the objective functions and constraint functions are monotone with respect to the uncertain parameters, the uncertain programming model may be converted to a crisp mathematical programming.

It is fortunate for us that almost all objective and constraint functions in practical problems are indeed monotone with respect to the uncertain parameters (not decision variables).

From the mathematical viewpoint, there is no difference between crisp mathematical programming and classical mathematical programming except for an integral. Thus we may solve it by simplex method, branch-and-bound method, cutting plane method, implicit enumeration method, interior point method, gradient method, genetic algorithm, particle swarm optimization, neural networks, tabu search, and so on.

Example 3.1: Assume that x_1, x_2, x_3 are nonnegative decision variables, ξ_1, ξ_2, ξ_3 are independent linear uncertain variables $\mathcal{L}(1,2), \mathcal{L}(2,3), \mathcal{L}(3,4)$, and η_1, η_2, η_3 are independent zigzag uncertain variables $\mathcal{Z}(1,2,3), \mathcal{Z}(2,3,4)$, $\mathcal{Z}(3,4,5)$, respectively. Consider the uncertain programming,

$$\begin{cases} \max\limits_{x_1, x_2, x_3} E\left[\sqrt{x_1 + \xi_1} + \sqrt{x_2 + \xi_2} + \sqrt{x_3 + \xi_3}\right] \\ \text{subject to:} \\ \qquad \mathcal{M}\{(x_1 + \eta_1)^2 + (x_2 + \eta_2)^2 + (x_3 + \eta_3)^2 \le 100\} \ge 0.9 \\ \qquad x_1, x_2, x_3 \ge 0. \end{cases}$$

Note that $\sqrt{x_1 + \xi_1} + \sqrt{x_2 + \xi_2} + \sqrt{x_3 + \xi_3}$ is a strictly increasing function with respect to ξ_1, ξ_2, ξ_3, and $(x_1 + \eta_1)^2 + (x_2 + \eta_2)^2 + (x_3 + \eta_3)^2$ is a strictly increasing function with respect to η_1, η_2, η_3. It is easy to verify that the uncertain programming model can be converted to the crisp model,

$$\begin{cases} \max\limits_{x_1,x_2,x_3} \displaystyle\int_0^1 \left(\sqrt{x_1+\Phi_1^{-1}(\alpha)}+\sqrt{x_2+\Phi_2^{-1}(\alpha)}+\sqrt{x_3+\Phi_3^{-1}(\alpha)}\right) \mathrm{d}\alpha \\ \text{subject to:} \\ \qquad (x_1+\Psi_1^{-1}(0.9))^2+(x_2+\Psi_2^{-1}(0.9))^2+(x_3+\Psi_3^{-1}(0.9))^2 \le 100 \\ \qquad x_1, x_2, x_3 \ge 0 \end{cases}$$

where $\Phi_1^{-1}, \Phi_2^{-1}, \Phi_3^{-1}, \Psi_1^{-1}, \Psi_2^{-1}, \Psi_3^{-1}$ are inverse uncertainty distributions of uncertain variables $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$, respectively. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may solve this model and obtain an optimal solution

$$(x_1^*, x_2^*, x_3^*) = (2.9735, 1.9735, 0.9735)$$

whose objective value is 6.3419.

Example 3.2: Assume that x_1 and x_2 are decision variables, ξ_1 and ξ_2 are iid linear uncertain variables $\mathcal{L}(0, \pi/2)$. Consider the uncertain programming,

$$\begin{cases} \min\limits_{x_1,x_2} E\left[x_1\sin(x_1-\xi_1)-x_2\cos(x_2+\xi_2)\right] \\ \text{subject to:} \\ \qquad 0 \le x_1 \le \dfrac{\pi}{2}, \quad 0 \le x_2 \le \dfrac{\pi}{2}. \end{cases}$$

It is clear that $x_1\sin(x_1-\xi_1)-x_2\cos(x_2+\xi_2)$ is strictly decreasing with respect to ξ_1 and strictly increasing with respect to ξ_2. Thus the uncertain programming is equivalent to the crisp model,

$$\begin{cases} \min\limits_{x_1,x_2} \displaystyle\int_0^1 \left(x_1\sin(x_1-\Phi_1^{-1}(1-\alpha))-x_2\cos(x_2+\Phi_2^{-1}(\alpha))\right) \mathrm{d}\alpha \\ \text{subject to:} \\ \qquad 0 \le x_1 \le \dfrac{\pi}{2}, \quad 0 \le x_2 \le \dfrac{\pi}{2} \end{cases}$$

where Φ_1^{-1}, Φ_2^{-1} are inverse uncertainty distributions of ξ_1, ξ_2, respectively. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may solve this model and obtain an optimal solution

$$(x_1^*, x_2^*) = (0.4026, 0.4026)$$

whose objective value is -0.2708.

3.3 Machine Scheduling Problem

Machine scheduling problem is concerned with finding an efficient schedule during an uninterrupted period of time for a set of machines to process a set of jobs. A lot of research work has been done on this type of problem. The study of machine scheduling problem with uncertain processing times was started by Liu [129] in 2010.

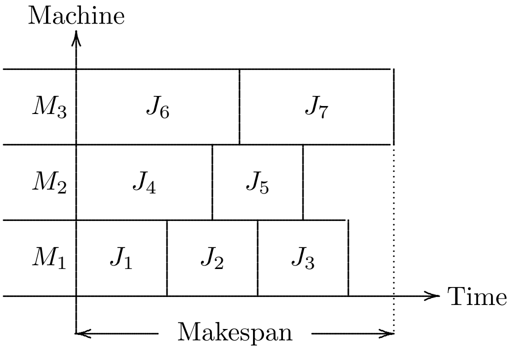

Figure 3.1: A Machine Schedule with 3 Machines and 7 Jobs. Reprinted from Liu [129].

In a machine scheduling problem, we assume that (a) each job can be processed on any machine without interruption; (b) each machine can process only one job at a time; and (c) the processing times are uncertain variables with known uncertainty distributions. We also use the following indices and parameters:

$i = 1, 2, \cdots, n$: jobs;
$k = 1, 2, \cdots, m$: machines;
ξ_{ik}: uncertain processing time of job i on machine k;
Φ_{ik}: uncertainty distribution of ξ_{ik}.

How to Represent a Schedule?

Liu [114] suggested that a schedule should be represented by two decision vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, where

$\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$: integer decision vector representing n jobs with $1 \le x_i \le n$ and $x_i \ne x_j$ for all $i \ne j$, $i, j = 1, 2, \cdots, n$. That is, the sequence $\{x_1, x_2, \cdots, x_n\}$ is a rearrangement of $\{1, 2, \cdots, n\}$;

$\boldsymbol{y} = (y_1, y_2, \cdots, y_{m-1})$: integer decision vector with $y_0 \equiv 0 \le y_1 \le y_2 \le \cdots \le y_{m-1} \le n \equiv y_m$.

We note that the schedule is fully determined by the decision vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in the following way. For each k $(1 \le k \le m)$, if $y_k = y_{k-1}$, then the machine k is not used; if $y_k > y_{k-1}$, then the machine k is used and processes jobs $x_{y_{k-1}+1}, x_{y_{k-1}+2}, \cdots, x_{y_k}$ in turn. Thus the schedule of all machines is as follows,

$$\begin{aligned}&\text{Machine 1: } x_{y_0+1} \to x_{y_0+2} \to \cdots \to x_{y_1};\\ &\text{Machine 2: } x_{y_1+1} \to x_{y_1+2} \to \cdots \to x_{y_2};\\ &\quad\cdots\\ &\text{Machine } m\text{: } x_{y_{m-1}+1} \to x_{y_{m-1}+2} \to \cdots \to x_{y_m}.\end{aligned} \tag{3.19}$$

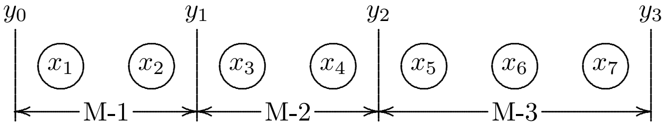

Figure 3.2: Formulation of Schedule in which Machine 1 processes Jobs x_1, x_2, Machine 2 processes Jobs x_3, x_4 and Machine 3 processes Jobs x_5, x_6, x_7. Reprinted from Liu [129].

Completion Times

Let $C_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi})$ be the completion times of jobs i, $i = 1, 2, \cdots, n$, respectively. For each k with $1 \le k \le m$, if the machine k is used (i.e., $y_k > y_{k-1}$), then we have

$$C_{x_{y_{k-1}+1}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi}) = \xi_{x_{y_{k-1}+1}k} \tag{3.20}$$

and

$$C_{x_{y_{k-1}+j}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi}) = C_{x_{y_{k-1}+j-1}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi}) + \xi_{x_{y_{k-1}+j}k} \tag{3.21}$$

for $2 \le j \le y_k - y_{k-1}$.

If the machine k is used, then the completion time $C_{x_{y_{k-1}+1}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi})$ of job $x_{y_{k-1}+1}$ is an uncertain variable whose inverse uncertainty distribution is

$$\Psi^{-1}_{x_{y_{k-1}+1}}(\boldsymbol{x}, \boldsymbol{y}, \alpha) = \Phi^{-1}_{x_{y_{k-1}+1}k}(\alpha). \tag{3.22}$$

Generally, suppose the completion time $C_{x_{y_{k-1}+j-1}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi})$ has an inverse uncertainty distribution $\Psi^{-1}_{x_{y_{k-1}+j-1}}(\boldsymbol{x}, \boldsymbol{y}, \alpha)$. Then the completion time $C_{x_{y_{k-1}+j}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi})$ has an inverse uncertainty distribution

$$\Psi^{-1}_{x_{y_{k-1}+j}}(\boldsymbol{x}, \boldsymbol{y}, \alpha) = \Psi^{-1}_{x_{y_{k-1}+j-1}}(\boldsymbol{x}, \boldsymbol{y}, \alpha) + \Phi^{-1}_{x_{y_{k-1}+j}k}(\alpha). \tag{3.23}$$

This recursive process may produce all inverse uncertainty distributions of completion times of jobs.

Makespan

Note that, for each k $(1 \le k \le m)$, the value $C_{x_{y_k}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi})$ is just the time that the machine k finishes all jobs assigned to it. Thus the makespan of the schedule $(\boldsymbol{x}, \boldsymbol{y})$ is determined by

$$f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi}) = \max_{1 \le k \le m} C_{x_{y_k}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi}) \tag{3.24}$$

whose inverse uncertainty distribution is

$$\Upsilon^{-1}(\boldsymbol{x}, \boldsymbol{y}, \alpha) = \max_{1 \le k \le m} \Psi^{-1}_{x_{y_k}}(\boldsymbol{x}, \boldsymbol{y}, \alpha). \tag{3.25}$$

Machine Scheduling Model

In order to minimize the expected makespan $E[f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi})]$, we have the following machine scheduling model,

$$\begin{cases} \min\limits_{\boldsymbol{x}, \boldsymbol{y}} E[f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi})] \\ \text{subject to:} \\ \qquad 1 \le x_i \le n, \quad i = 1, 2, \cdots, n \\ \qquad x_i \ne x_j, \quad i \ne j, \ i, j = 1, 2, \cdots, n \\ \qquad 0 \le y_1 \le y_2 \cdots \le y_{m-1} \le n \\ \qquad x_i, y_j, \quad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m-1, \quad \text{integers.} \end{cases} \tag{3.26}$$

Since $\Upsilon^{-1}(\boldsymbol{x}, \boldsymbol{y}, \alpha)$ is the inverse uncertainty distribution of $f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi})$, the machine scheduling model is simplified as follows,

$$\begin{cases} \min\limits_{\boldsymbol{x}, \boldsymbol{y}} \displaystyle\int_0^1 \Upsilon^{-1}(\boldsymbol{x}, \boldsymbol{y}, \alpha) \mathrm{d}\alpha \\ \text{subject to:} \\ \qquad 1 \le x_i \le n, \quad i = 1, 2, \cdots, n \\ \qquad x_i \ne x_j, \quad i \ne j, \ i, j = 1, 2, \cdots, n \\ \qquad 0 \le y_1 \le y_2 \cdots \le y_{m-1} \le n \\ \qquad x_i, y_j, \quad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m-1, \quad \text{integers.} \end{cases} \tag{3.27}$$

Numerical Experiment

Assume that there are 3 machines and 7 jobs with the following linear uncertain processing times

$$\xi_{ik} \sim \mathcal{L}(i, i+k), \quad i = 1, 2, \cdots, 7, \ k = 1, 2, 3$$

where i is the index of jobs and k is the index of machines. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields that the optimal solution is

$$\boldsymbol{x}^* = (1, 4, 5, 3, 7, 2, 6), \quad \boldsymbol{y}^* = (3, 5). \tag{3.28}$$

In other words, the optimal machine schedule is

Machine 1: $1 \to 4 \to 5$
Machine 2: $3 \to 7$
Machine 3: $2 \to 6$

whose expected makespan is 12.

3.4 Vehicle Routing Problem

Vehicle routing problem (VRP) is concerned with finding efficient routes, beginning and ending at a central depot, for a fleet of vehicles to serve a number of customers.

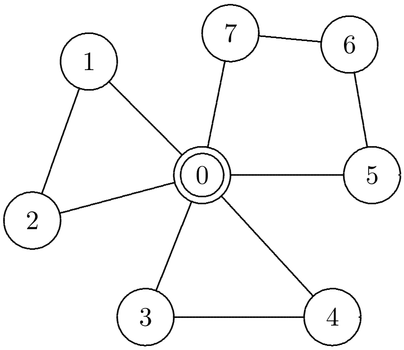

Figure 3.3: A Vehicle Routing Plan with Single Depot and 7 Customers. Reprinted from Liu [129].

Due to its wide applicability and economic importance, vehicle routing problem has been extensively studied. Liu [129] first introduced uncertainty theory into the research area of vehicle routing problem in 2010. In this section, vehicle routing problem will be modelled by uncertain programming in which the travel times are assumed to be uncertain variables with known uncertainty distributions.

We assume that (a) a vehicle will be assigned for only one route on which there may be more than one customer; (b) a customer will be visited by one and only one vehicle; (c) each route begins and ends at the depot; and (d) each customer specifies its time window within which the delivery is permitted or preferred to start.

Let us first introduce the following indices and model parameters:

$i = 0$: depot;

$i = 1, 2, \cdots, n$: customers;
$k = 1, 2, \cdots, m$: vehicles;
D_{ij}: travel distance from customers i to j, $i, j = 0, 1, 2, \cdots, n$;
T_{ij}: uncertain travel time from customers i to j, $i, j = 0, 1, 2, \cdots, n$;
Φ_{ij}: uncertainty distribution of T_{ij}, $i, j = 0, 1, 2, \cdots, n$;
$[a_i, b_i]$: time window of customer i, $i = 1, 2, \cdots, n$.

Operational Plan

Liu [114] suggested that an operational plan should be represented by three decision vectors $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{t}$, where

$\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$: integer decision vector representing n customers with $1 \le x_i \le n$ and $x_i \ne x_j$ for all $i \ne j$, $i, j = 1, 2, \cdots, n$. That is, the sequence $\{x_1, x_2, \cdots, x_n\}$ is a rearrangement of $\{1, 2, \cdots, n\}$;

$\boldsymbol{y} = (y_1, y_2, \cdots, y_{m-1})$: integer decision vector with $y_0 \equiv 0 \le y_1 \le y_2 \le \cdots \le y_{m-1} \le n \equiv y_m$;

$\boldsymbol{t} = (t_1, t_2, \cdots, t_m)$: each t_k represents the starting time of vehicle k at the depot, $k = 1, 2, \cdots, m$.

We note that the operational plan is fully determined by the decision vectors $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{t}$ in the following way. For each k $(1 \le k \le m)$, if $y_k = y_{k-1}$, then vehicle k is not used; if $y_k > y_{k-1}$, then vehicle k is used and starts from the depot at time t_k, and the tour of vehicle k is $0 \to x_{y_{k-1}+1} \to x_{y_{k-1}+2} \to \cdots \to x_{y_k} \to 0$. Thus the tours of all vehicles are as follows:

$$\begin{aligned}
&\text{Vehicle 1: } 0 \to x_{y_0+1} \to x_{y_0+2} \to \cdots \to x_{y_1} \to 0;\\
&\text{Vehicle 2: } 0 \to x_{y_1+1} \to x_{y_1+2} \to \cdots \to x_{y_2} \to 0;\\
&\qquad \cdots\\
&\text{Vehicle } m\text{: } 0 \to x_{y_{m-1}+1} \to x_{y_{m-1}+2} \to \cdots \to x_{y_m} \to 0.
\end{aligned}$$

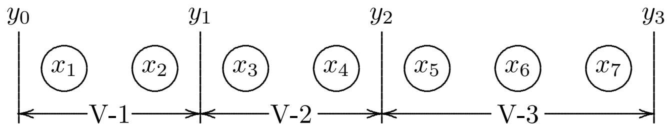

Figure 3.4: Formulation of Operational Plan in which Vehicle 1 visits Customers x_1, x_2, Vehicle 2 visits Customers x_3, x_4 and Vehicle 3 visits Customers x_5, x_6, x_7. Reprinted from Liu [129].

It is clear that this type of representation is intuitive, and the total number of decision variables is $n + 2m - 1$. We also note that the above decision variables $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{t}$ ensure that: (a) each vehicle will be used at most one time; (b) all tours begin and end at the depot; (c) each customer will be visited by one and only one vehicle; and (d) there is no subtour.

Arrival Times

Let $f_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t})$ be the arrival time function of some vehicles at customers i for $i = 1, 2, \cdots, n$. We remind readers that $f_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t})$ are determined by the decision variables $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{t}$, $i = 1, 2, \cdots, n$. Since unloading can start either immediately, or later, when a vehicle arrives at a customer, the calculation of $f_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t})$ is heavily dependent on the operational strategy. Here we assume that the customer does not permit a delivery earlier than the time window. That is, the vehicle will wait to unload until the beginning of the time window if it arrives before the time window. If a vehicle arrives at a customer after the beginning of the time window, unloading will start immediately. For each k with $1 \le k \le m$, if vehicle k is used (i.e., $y_k > y_{k-1}$), then we have

$$f_{x_{y_{k-1}+1}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}) = t_k + T_{0x_{y_{k-1}+1}}$$

and

$$f_{x_{y_{k-1}+j}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}) = f_{x_{y_{k-1}+j-1}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}) \vee a_{x_{y_{k-1}+j-1}} + T_{x_{y_{k-1}+j-1}x_{y_{k-1}+j}}$$

for $2 \le j \le y_k - y_{k-1}$. If the vehicle k is used, i.e., $y_k > y_{k-1}$, then the arrival time $f_{x_{y_{k-1}+1}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t})$ at the customer $x_{y_{k-1}+1}$ is an uncertain variable whose inverse uncertainty distribution is

$$\Psi^{-1}_{x_{y_{k-1}+1}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}, \alpha) = t_k + \Phi^{-1}_{0x_{y_{k-1}+1}}(\alpha).$$

Generally, suppose the arrival time $f_{x_{y_{k-1}+j-1}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t})$ has an inverse uncertainty distribution $\Psi^{-1}_{x_{y_{k-1}+j-1}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}, \alpha)$. Then $f_{x_{y_{k-1}+j}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t})$ has an inverse uncertainty distribution

$$\Psi^{-1}_{x_{y_{k-1}+j}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}, \alpha) = \Psi^{-1}_{x_{y_{k-1}+j-1}}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}, \alpha) \vee a_{x_{y_{k-1}+j-1}} + \Phi^{-1}_{x_{y_{k-1}+j-1}x_{y_{k-1}+j}}(\alpha)$$

for $2 \le j \le y_k - y_{k-1}$. This recursive process may produce all inverse uncertainty distributions of arrival times at customers.

Travel Distance

Let $g(\boldsymbol{x}, \boldsymbol{y})$ be the total travel distance of all vehicles. Then we have

$$g(\boldsymbol{x}, \boldsymbol{y}) = \sum_{k=1}^{m} g_k(\boldsymbol{x}, \boldsymbol{y}) \tag{3.29}$$

where

$$g_k(\boldsymbol{x}, \boldsymbol{y}) = \begin{cases} D_{0x_{y_{k-1}+1}} + \sum\limits_{j=y_{k-1}+1}^{y_k-1} D_{x_j x_{j+1}} + D_{x_{y_k}0}, & \text{if } y_k > y_{k-1} \\ 0, & \text{if } y_k = y_{k-1} \end{cases}$$

for $k = 1, 2, \cdots, m$.

Vehicle Routing Model

If we hope that each customer i $(1 \le i \le n)$ is visited within its time window $[a_i, b_i]$ with confidence level α_i (i.e., the vehicle arrives at customer i before time b_i), then we have the following chance constraint,

$$\mathcal{M}\{f_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}) \le b_i\} \ge \alpha_i. \tag{3.30}$$

If we want to minimize the total travel distance of all vehicles subject to the time window constraint, then we have the following vehicle routing model,

$$\left\{\begin{array}{l} \min\limits_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}} g(\boldsymbol{x}, \boldsymbol{y}) \\ \text{subject to:} \\ \qquad \mathcal{M}\{f_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}) \le b_i\} \ge \alpha_i, \quad i = 1, 2, \cdots, n \\ \qquad 1 \le x_i \le n, \quad i = 1, 2, \cdots, n \\ \qquad x_i \ne x_j, \quad i \ne j, \ i, j = 1, 2, \cdots, n \\ \qquad 0 \le y_1 \le y_2 \le \cdots \le y_{m-1} \le n \\ \qquad x_i, y_j, \quad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m-1, \quad \text{integers} \end{array}\right. \tag{3.31}$$

which is equivalent to

$$\left\{\begin{array}{l} \min\limits_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{t}} g(\boldsymbol{x}, \boldsymbol{y}) \\ \text{subject to:} \\ \qquad \Psi_i^{-1}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}, \alpha_i) \le b_i, \quad i = 1, 2, \cdots, n \\ \qquad 1 \le x_i \le n, \quad i = 1, 2, \cdots, n \\ \qquad x_i \ne x_j, \quad i \ne j, \ i, j = 1, 2, \cdots, n \\ \qquad 0 \le y_1 \le y_2 \le \cdots \le y_{m-1} \le n \\ \qquad x_i, y_j, \quad i = 1, 2, \cdots, n, \quad j = 1, 2, \cdots, m-1, \quad \text{integers} \end{array}\right. \tag{3.32}$$

where $\Psi_i^{-1}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}, \alpha)$ are the inverse uncertainty distributions of $f_i(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t})$ for $i = 1, 2, \cdots, n$, respectively.

Numerical Experiment

Assume that there are 3 vehicles and 7 customers with time windows shown in Table 3.1, and each customer is visited within time windows with confidence level 0.90.

We also assume that the distances are $D_{ij} = |i - j|$ for $i, j = 0, 1, 2, \cdots, 7$, and the travel times are normal uncertain variables

$$T_{ij} \sim \mathcal{N}(2|i - j|, 1), \quad i, j = 0, 1, 2, \cdots, 7.$$

The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may

Table 3.1: Time Windows of Customers

Node	Window	Node	Window
1	$[7:00, 9:00]$	5	$[15:00, 17:00]$
2	$[7:00, 9:00]$	6	$[19:00, 21:00]$
3	$[15:00, 17:00]$	7	$[19:00, 21:00]$
4	$[15:00, 17:00]$		

yield that the optimal solution is

$$\begin{aligned} \boldsymbol{x}^* &= (1, 3, 2, 5, 7, 4, 6), \\ \boldsymbol{y}^* &= (2, 5), \\ \boldsymbol{t}^* &= (6:18, 4:18, 8:18). \end{aligned} \tag{3.33}$$

In other words, the optimal operational plan is

Vehicle 1: depot $\to 1 \to 3 \to$ depot (the latest starting time is 6:18)
Vehicle 2: depot $\to 2 \to 5 \to 7 \to$ depot (the latest starting time is 4:18)
Vehicle 3: depot $\to 4 \to 6 \to$ depot (the latest starting time is 8:18)

whose total travel distance is 32.

3.5 Project Scheduling Problem

Project scheduling problem is to determine the schedule of allocating resources so as to balance the total cost and the completion time. The study of project scheduling problem with uncertain factors was started by Liu [129] in 2010. This section presents an uncertain programming model for project scheduling problem in which the duration times are assumed to be uncertain variables with known uncertainty distributions.

Project scheduling is usually represented by a directed acyclic network where nodes correspond to milestones, and arcs to activities which are basically characterized by the times and costs consumed.

Let $(\mathcal{V}, \mathcal{A})$ be a directed acyclic graph, where $\mathcal{V} = \{1, 2, \cdots, n, n+1\}$ is the set of nodes, $\mathcal{A}$ is the set of arcs, $(i, j) \in \mathcal{A}$ is the arc of the graph $(\mathcal{V}, \mathcal{A})$ from nodes i to j. It is well-known that we can rearrange the indexes of the nodes in $\mathcal{V}$ such that $i < j$ for all $(i, j) \in \mathcal{A}$.

Before we begin to study project scheduling problem with uncertain activity duration times, we first make some assumptions: (a) all of the costs needed are obtained via loans with some given interest rate; and (b) each activity can be processed only if the loan needed is allocated and all the foregoing activities are finished.

In order to model the project scheduling problem, we introduce the following indices and parameters:

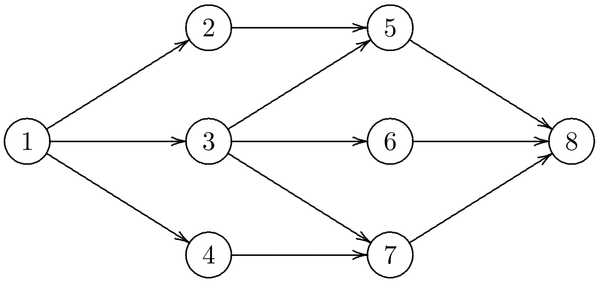

Figure 3.5: A Project with 8 Milestones and 11 Activities. Reprinted from Liu [129].

ξ_{ij}: uncertain duration time of activity (i,j) in $\mathcal{A}$;
Φ_{ij}: uncertainty distribution of ξ_{ij};
c_{ij}: cost of activity (i,j) in $\mathcal{A}$;
r: interest rate;
x_i: integer decision variable representing the allocating time of all loans needed for all activities (i,j) in $\mathcal{A}$.

Starting Times

For simplicity, we write $\boldsymbol{\xi} = \{\xi_{ij} : (i,j) \in \mathcal{A}\}$ and $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$. Let $T_i(\boldsymbol{x}, \boldsymbol{\xi})$ denote the starting time of all activities (i,j) in $\mathcal{A}$. According to the assumptions, the starting time of the total project (i.e., the starting time of of all activities $(1,j)$ in $\mathcal{A}$) should be

$$T_1(\boldsymbol{x}, \boldsymbol{\xi}) = x_1 \tag{3.34}$$

whose inverse uncertainty distribution may be written as

$$\Psi_1^{-1}(\boldsymbol{x}, \alpha) = x_1. \tag{3.35}$$

From the starting time $T_1(\boldsymbol{x}, \boldsymbol{\xi})$, we deduce that the starting time of activity $(2,5)$ is

$$T_2(\boldsymbol{x}, \boldsymbol{\xi}) = x_2 \vee (x_1 + \xi_{12}) \tag{3.36}$$

whose inverse uncertainty distribution may be written as

$$\Psi_2^{-1}(\boldsymbol{x}, \alpha) = x_2 \vee (x_1 + \Phi_{12}^{-1}(\alpha)). \tag{3.37}$$

Generally, suppose that the starting time $T_k(\boldsymbol{x}, \boldsymbol{\xi})$ of all activities (k,i) in $\mathcal{A}$ has an inverse uncertainty distribution $\Psi_k^{-1}(\boldsymbol{x}, \alpha)$. Then the starting time $T_i(\boldsymbol{x}, \boldsymbol{\xi})$ of all activities (i,j) in $\mathcal{A}$ should be

$$T_i(\boldsymbol{x}, \boldsymbol{\xi}) = x_i \vee \max_{(k,i)\in\mathcal{A}} (T_k(\boldsymbol{x}, \boldsymbol{\xi}) + \xi_{ki}) \tag{3.38}$$

whose inverse uncertainty distribution is

$$\Psi_i^{-1}(\boldsymbol{x}, \alpha) = x_i \vee \max_{(k,i)\in\mathcal{A}} \left(\Psi_k^{-1}(\boldsymbol{x}, \alpha) + \Phi_{ki}^{-1}(\alpha)\right). \tag{3.39}$$

This recursive process may produce all inverse uncertainty distributions of starting times of activities.

Completion Time

The completion time $T(\boldsymbol{x}, \boldsymbol{\xi})$ of the total project (i.e, the finish time of all activities $(k, n+1)$ in $\mathcal{A}$) is

$$T(\boldsymbol{x}, \boldsymbol{\xi}) = \max_{(k,n+1)\in\mathcal{A}} (T_k(\boldsymbol{x}, \boldsymbol{\xi}) + \xi_{k,n+1}) \tag{3.40}$$

whose inverse uncertainty distribution is

$$\Psi^{-1}(\boldsymbol{x}, \alpha) = \max_{(k,n+1)\in\mathcal{A}} \left(\Psi_k^{-1}(\boldsymbol{x}, \alpha) + \Phi_{k,n+1}^{-1}(\alpha)\right). \tag{3.41}$$

Total Cost

Based on the completion time $T(\boldsymbol{x}, \boldsymbol{\xi})$, the total cost of the project can be written as

$$C(\boldsymbol{x}, \boldsymbol{\xi}) = \sum_{(i,j)\in\mathcal{A}} c_{ij} (1+r)^{\lceil T(\boldsymbol{x},\boldsymbol{\xi}) - x_i \rceil} \tag{3.42}$$

where $\lceil a \rceil$ represents the minimal integer greater than or equal to a. Note that $C(\boldsymbol{x}, \boldsymbol{\xi})$ is a discrete uncertain variable whose inverse uncertainty distribution is

$$\Upsilon^{-1}(\boldsymbol{x}, \alpha) = \sum_{(i,j)\in\mathcal{A}} c_{ij} (1+r)^{\lceil \Psi^{-1}(\boldsymbol{x};\alpha) - x_i \rceil} \tag{3.43}$$

for $0 < \alpha < 1$.

Project Scheduling Model

In order to minimize the expected cost of the project under the completion time constraint, we may construct the following project scheduling model,

$$\begin{cases} \min\limits_{\boldsymbol{x}} E[C(\boldsymbol{x}, \boldsymbol{\xi})] \\ \text{subject to:} \\ \qquad \mathcal{M}\{T(\boldsymbol{x}, \boldsymbol{\xi}) \le T_0\} \ge \alpha_0 \\ \qquad \boldsymbol{x} \ge 0, \text{ integer vector} \end{cases} \tag{3.44}$$

where T_0 is a due date of the project, α_0 is a predetermined confidence level, $T(\boldsymbol{x}, \boldsymbol{\xi})$ is the completion time defined by (3.40), and $C(\boldsymbol{x}, \boldsymbol{\xi})$ is the total cost defined by (3.42). This model is equivalent to

$$\begin{cases} \min\limits_{\boldsymbol{x}} \int_0^1 \Upsilon^{-1}(\boldsymbol{x}, \alpha)\mathrm{d}\alpha \\ \text{subject to:} \\ \qquad \Psi^{-1}(\boldsymbol{x}, \alpha_0) \le T_0 \\ \qquad \boldsymbol{x} \ge 0, \text{ integer vector} \end{cases} \tag{3.45}$$

where $\Psi^{-1}(\boldsymbol{x}, \alpha)$ is the inverse uncertainty distribution of $T(\boldsymbol{x}, \boldsymbol{\xi})$ determined by (3.41) and $\Upsilon^{-1}(\boldsymbol{x}, \alpha)$ is the inverse uncertainty distribution of $C(\boldsymbol{x}, \boldsymbol{\xi})$ determined by (3.43).

Numerical Experiment

Consider a project scheduling problem shown by Figure 3.5 in which there are 8 milestones and 11 activities. Assume that all duration times of activities are linear uncertain variables,

$$\xi_{ij} \sim \mathcal{L}(3i, 3j), \quad \forall (i,j) \in \mathcal{A}$$

and assume that the costs of activities are

$$c_{ij} = i + j, \quad \forall (i,j) \in \mathcal{A}.$$

In addition, we also suppose that the interest rate is $r = 0.02$, the due date is $T_0 = 60$, and the confidence level is $\alpha_0 = 0.85$. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields that the optimal solution is

$$\boldsymbol{x}^* = (7, 24, 17, 16, 35, 33, 30). \tag{3.46}$$

In other words, the optimal allocating times of all loans needed for all activities are shown in Table 3.2 whose expected total cost is 190.6, and

$$\mathcal{M}\{T(\boldsymbol{x}^*, \boldsymbol{\xi}) \le 60\} = 0.88.$$

Table 3.2: Optimal Allocating Times of Loans

Date	7	16	17	24	30	33	35
Node	1	4	3	2	7	6	5
Loan	12	11	27	7	15	14	13

3.6 Uncertain Multiobjective Programming

It has been increasingly recognized that many real decision-making problems involve multiple, noncommensurable, and conflicting objectives which should be considered simultaneously. In order to optimize multiple objectives, multiobjective programming has been well developed and applied widely. For modelling multiobjective decision-making problems with uncertain parameters, Liu and Chen [141] presented the following uncertain multiobjective programming,

$$\begin{cases} \min\limits_{\boldsymbol{x}} \left(E[f_1(\boldsymbol{x},\boldsymbol{\xi})], E[f_2(\boldsymbol{x},\boldsymbol{\xi})], \cdots, E[f_m(\boldsymbol{x},\boldsymbol{\xi})]\right) \\ \text{subject to:} \\ \qquad \mathcal{M}\{g_j(\boldsymbol{x},\boldsymbol{\xi}) \le 0\} \ge \alpha_j, \quad j = 1,2,\cdots,p \end{cases} \tag{3.47}$$

where $f_i(\boldsymbol{x},\boldsymbol{\xi})$ are objective functions for $i = 1,2,\cdots,m$, and $g_j(\boldsymbol{x},\boldsymbol{\xi})$ are constraint functions for $j = 1,2,\cdots,p$.

Since the objectives are usually in conflict, there is no optimal solution that simultaneously minimizes all the objective functions. In this case, we have to introduce the concept of *Pareto solution*, which means that it is impossible to improve any one objective without sacrificing on one or more of the other objectives.

Definition 3.3 *A feasible solution $\boldsymbol{x}^*$ is said to be Pareto to the uncertain multiobjective programming (3.47) if there is no feasible solution $\boldsymbol{x}$ such that*

$$E[f_i(\boldsymbol{x},\boldsymbol{\xi})] \le E[f_i(\boldsymbol{x}^*,\boldsymbol{\xi})], \quad i = 1,2,\cdots,m \tag{3.48}$$

and $E[f_j(\boldsymbol{x},\boldsymbol{\xi})] < E[f_j(\boldsymbol{x}^*,\boldsymbol{\xi})]$ for at least one index j.

If the decision maker has a real-valued *preference function* aggregating the m objective functions, then we may minimize the aggregating preference function subject to the same set of chance constraints. This model is referred to as a *compromise model* whose solution is called a *compromise solution*. It has been proved that the compromise solution is Pareto to the original multiobjective model.

The first well-known compromise model is set up by weighting the objective functions, i.e.,

$$\begin{cases} \min\limits_{\boldsymbol{x}} \sum\limits_{i=1}^{m} \lambda_i E[f_i(\boldsymbol{x},\boldsymbol{\xi})] \\ \text{subject to:} \\ \qquad \mathcal{M}\{g_j(\boldsymbol{x},\boldsymbol{\xi}) \le 0\} \ge \alpha_j, \quad j = 1,2,\cdots,p \end{cases} \tag{3.49}$$

where the weights $\lambda_1,\lambda_2,\cdots,\lambda_m$ are nonnegative numbers with $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$, for example, $\lambda_i \equiv 1/m$ for $i = 1,2,\cdots,m$.

The second way is related to minimizing the *distance function* from a solution

$$(E[f_1(\boldsymbol{x},\boldsymbol{\xi})], E[f_2(\boldsymbol{x},\boldsymbol{\xi})], \cdots, E[f_m(\boldsymbol{x},\boldsymbol{\xi})]) \tag{3.50}$$

to an ideal vector $(f_1^*, f_2^*, \cdots, f_m^*)$, where f_i^* are the optimal values of the ith objective functions without considering other objectives, $i = 1, 2, \cdots, m$, respectively. That is,

$$\begin{cases} \min\limits_{\boldsymbol{x}} \sum\limits_{i=1}^{m} \lambda_i (E[f_i(\boldsymbol{x},\boldsymbol{\xi})] - f_i^*)^2 \\ \text{subject to:} \\ \qquad \mathcal{M}\{g_j(\boldsymbol{x},\boldsymbol{\xi}) \le 0\} \ge \alpha_j, \quad j = 1, 2, \cdots, p \end{cases} \tag{3.51}$$

where the weights $\lambda_1, \lambda_2, \cdots, \lambda_m$ are nonnegative numbers with $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$, for example, $\lambda_i \equiv 1/m$ for $i = 1, 2, \cdots, m$.

By the third way a compromise solution can be found via an *interactive approach* consisting of a sequence of decision phases and computation phases. Various interactive approaches have been developed.

3.7 Uncertain Goal Programming

The concept of goal programming was presented by Charnes and Cooper [11] in 1961 and subsequently studied by many researchers. Goal programming can be regarded as a special compromise model for multiobjective optimization and has been applied in a wide variety of real-world problems. In multiobjective decision-making problems, we assume that the decision-maker is able to assign a target level for each goal and the key idea is to minimize the deviations (positive, negative, or both) from the target levels. In the real-world situation, the goals are achievable only at the expense of other goals and these goals are usually incompatible. In order to balance multiple conflicting objectives, a decision-maker may establish a hierarchy of importance among these incompatible goals so as to satisfy as many goals as possible in the order specified. For multiobjective decision-making problems with uncertain parameters, Liu and Chen [141] proposed an uncertain goal programming,

$$\begin{cases} \min\limits_{\boldsymbol{x}} \sum\limits_{j=1}^{l} P_j \sum\limits_{i=1}^{m} (u_{ij} d_i^+ + v_{ij} d_i^-) \\ \text{subject to:} \\ \qquad E[f_i(\boldsymbol{x},\boldsymbol{\xi})] + d_i^- - d_i^+ = b_i, \quad i = 1, 2, \cdots, m \\ \qquad \mathcal{M}\{g_j(\boldsymbol{x},\boldsymbol{\xi}) \le 0\} \ge \alpha_j, \qquad j = 1, 2, \cdots, p \\ \qquad d_i^+, d_i^- \ge 0, \qquad\qquad\qquad\quad i = 1, 2, \cdots, m \end{cases} \tag{3.52}$$

where P_j is the preemptive priority factor which expresses the relative importance of various goals, $P_j \gg P_{j+1}$, for all j, u_{ij} is the weighting factor corresponding to positive deviation for goal i with priority j assigned, v_{ij} is the weighting factor corresponding to negative deviation for goal i with priority j assigned, d_i^+ is the positive deviation from the target of goal i, d_i^- is the negative deviation from the target of goal i, f_i is a function in goal constraints, g_j is a function in real constraints, b_i is the target value according to goal i, l is the number of priorities, m is the number of goal constraints, and p is the number of real constraints. Note that the positive and negative deviations are calculated by

$$d_i^+ = \begin{cases} E[f_i(\boldsymbol{x}, \boldsymbol{\xi})] - b_i, & \text{if } E[f_i(\boldsymbol{x}, \boldsymbol{\xi})] > b_i \\ 0, & \text{otherwise} \end{cases} \tag{3.53}$$

and

$$d_i^- = \begin{cases} b_i - E[f_i(\boldsymbol{x}, \boldsymbol{\xi})], & \text{if } E[f_i(\boldsymbol{x}, \boldsymbol{\xi})] < b_i \\ 0, & \text{otherwise} \end{cases} \tag{3.54}$$

for each i. Sometimes, the objective function in the goal programming model is written as follows,

$$\text{lexmin}\left\{\sum_{i=1}^m (u_{i1}d_i^+ + v_{i1}d_i^-), \sum_{i=1}^m (u_{i2}d_i^+ + v_{i2}d_i^-), \cdots, \sum_{i=1}^m (u_{il}d_i^+ + v_{il}d_i^-)\right\}$$

where lexmin represents lexicographically minimizing the objective vector.

3.8 Uncertain Multilevel Programming

Multilevel programming offers a means of studying decentralized decision systems in which we assume that the leader and followers may have their own decision variables and objective functions, and the leader can only influence the reactions of followers through his own decision variables, while the followers have full authority to decide how to optimize their own objective functions in view of the decisions of the leader and other followers.

Assume that in a decentralized two-level decision system there is one leader and m followers. Let $\boldsymbol{x}$ and $\boldsymbol{y}_i$ be the control vectors of the leader and the ith followers, $i = 1, 2, \cdots, m$, respectively. We also assume that the objective functions of the leader and ith followers are $F(\boldsymbol{x}, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_m, \boldsymbol{\xi})$ and $f_i(\boldsymbol{x}, \boldsymbol{y}_1, \cdots, \boldsymbol{y}_m, \boldsymbol{\xi})$, $i = 1, 2, \cdots, m$, respectively, where $\boldsymbol{\xi}$ is an uncertain vector.

Let the feasible set of control vector $\boldsymbol{x}$ of the leader be defined by the chance constraint

$$\mathcal{M}\{G(\boldsymbol{x}, \boldsymbol{\xi}) \le 0\} \ge \alpha \tag{3.55}$$

where G is a constraint function, and α is a predetermined confidence level. Then for each decision $\boldsymbol{x}$ chosen by the leader, the feasibility of control vectors $\boldsymbol{y}_i$ of the ith followers should be dependent on not only $\boldsymbol{x}$ but also

$\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{i-1}, \boldsymbol{y}_{i+1}, \cdots, \boldsymbol{y}_m$, and generally represented by the chance constraints,

$$\mathcal{M}\{g_i(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_m, \boldsymbol{\xi}) \le 0\} \ge \alpha_i \tag{3.56}$$

where g_i are constraint functions, and α_i are predetermined confidence levels, $i = 1, 2, \cdots, m$, respectively.

Assume that the leader first chooses his control vector $\boldsymbol{x}$, and the followers determine their control array $(\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_m)$ after that. In order to minimize the expected objective of the leader, Liu and Yao [140] proposed the following uncertain multilevel programming,

$$\begin{cases} \min\limits_{\boldsymbol{x}} E[F(\boldsymbol{x}, \boldsymbol{y}_1^*, \boldsymbol{y}_2^*, \cdots, \boldsymbol{y}_m^*, \boldsymbol{\xi})] \\ \text{subject to:} \\ \qquad \mathcal{M}\{G(\boldsymbol{x}, \boldsymbol{\xi}) \le 0\} \ge \alpha \\ \qquad (\boldsymbol{y}_1^*, \boldsymbol{y}_2^*, \cdots, \boldsymbol{y}_m^*) \text{ solves problems } (i = 1, 2, \cdots, m) \\ \qquad \begin{cases} \min\limits_{\boldsymbol{y}_i} E[f_i(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_m, \boldsymbol{\xi})] \\ \text{subject to:} \\ \qquad \mathcal{M}\{g_i(\boldsymbol{x}, \boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_m, \boldsymbol{\xi}) \le 0\} \ge \alpha_i. \end{cases} \end{cases} \tag{3.57}$$

Definition 3.4 *Let $\boldsymbol{x}$ be a feasible control vector of the leader. A Nash equilibrium of followers is the feasible array $(\boldsymbol{y}_1^*, \boldsymbol{y}_2^*, \cdots, \boldsymbol{y}_m^*)$ with respect to $\boldsymbol{x}$ if*

$$\begin{aligned} &E[f_i(\boldsymbol{x}, \boldsymbol{y}_1^*, \cdots, \boldsymbol{y}_{i-1}^*, \boldsymbol{y}_i, \boldsymbol{y}_{i+1}^*, \cdots, \boldsymbol{y}_m^*, \boldsymbol{\xi})] \\ &\qquad \ge E[f_i(\boldsymbol{x}, \boldsymbol{y}_1^*, \cdots, \boldsymbol{y}_{i-1}^*, \boldsymbol{y}_i^*, \boldsymbol{y}_{i+1}^*, \cdots, \boldsymbol{y}_m^*, \boldsymbol{\xi})] \end{aligned} \tag{3.58}$$

for any feasible array $(\boldsymbol{y}_1^*, \cdots, \boldsymbol{y}_{i-1}^*, \boldsymbol{y}_i, \boldsymbol{y}_{i+1}^*, \cdots, \boldsymbol{y}_m^*)$ and $i = 1, 2, \cdots, m$.

Definition 3.5 *Suppose that $\boldsymbol{x}^*$ is a feasible control vector of the leader and $(\boldsymbol{y}_1^*, \boldsymbol{y}_2^*, \cdots, \boldsymbol{y}_m^*)$ is a Nash equilibrium of followers with respect to $\boldsymbol{x}^*$. We call the array $(\boldsymbol{x}^*, \boldsymbol{y}_1^*, \boldsymbol{y}_2^*, \cdots, \boldsymbol{y}_m^*)$ a Stackelberg-Nash equilibrium to the uncertain multilevel programming (3.57) if*

$$E[F(\overline{\boldsymbol{x}}, \overline{\boldsymbol{y}}_1, \overline{\boldsymbol{y}}_2, \cdots, \overline{\boldsymbol{y}}_m, \boldsymbol{\xi})] \ge E[F(\boldsymbol{x}^*, \boldsymbol{y}_1^*, \boldsymbol{y}_2^*, \cdots, \boldsymbol{y}_m^*, \boldsymbol{\xi})] \tag{3.59}$$

for any feasible control vector $\overline{\boldsymbol{x}}$ and the Nash equilibrium $(\overline{\boldsymbol{y}}_1, \overline{\boldsymbol{y}}_2, \cdots, \overline{\boldsymbol{y}}_m)$ with respect to $\overline{\boldsymbol{x}}$.

3.9 Bibliographic Notes

Uncertain programming was founded by Liu [124] in 2009 and was applied to machine scheduling problem, vehicle routing problem and project scheduling problem by Liu [129] in 2010.

As extensions of uncertain programming theory, Liu and Chen [141] developed an uncertain multiobjective programming and an uncertain goal programming. In addition, Liu and Yao [140] suggested an uncertain multilevel programming for modeling decentralized decision systems with uncertain factors.

After that, the uncertain programming has obtained fruitful results in both theory and practice. For exploring more books and papers, the interested reader may visit the website at http://orsc.edu.cn/online.

Chapter 4

Uncertain Statistics

The study of uncertain statistics was started by Liu [129] in 2010. It is a methodology for collecting and interpreting expert's experimental data by uncertainty theory. This chapter will design a questionnaire survey for collecting expert's experimental data, and introduce empirical uncertainty distribution (i.e., linear interpolation method), principle of least squares, method of moments, and Delphi method for determining uncertainty distributions from expert's experimental data.

4.1 Expert's Experimental Data

Uncertain statistics is based on expert's experimental data rather than historical data. How do we obtain expert's experimental data? Liu [129] proposed a questionnaire survey for collecting expert's experimental data. The starting point is to invite one or more domain experts who are asked to complete a questionnaire about the meaning of an uncertain variable ξ like "how far from Beijing to Tianjin".

We first ask the domain expert to choose a possible value x (say 110km) that the uncertain variable ξ may take, and then quiz him

$$\text{"How likely is } \xi \text{ less than or equal to } x\text{?"} \tag{4.1}$$

Denote the expert's belief degree by α (say 0.6). Note that the expert's belief degree of ξ greater than x must be $1-\alpha$ due to the self-duality of uncertain measure. An expert's experimental data

$$(x, \alpha) = (110, 0.6) \tag{4.2}$$

is thus acquired from the domain expert.

Repeating the above process, the following expert's experimental data are obtained by the questionnaire,

$$(x_1, \alpha_1),\ (x_2, \alpha_2),\ \cdots,\ (x_n, \alpha_n). \tag{4.3}$$

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5_5

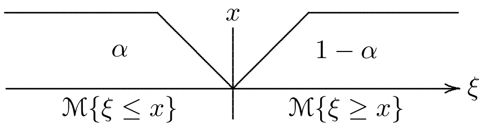

Figure 4.1: Expert's Experimental Data (x, α). Reprinted from Liu [129].

Remark 4.1: None of x, α and n could be assigned a value in the questionnaire before asking the domain expert. Otherwise, the domain expert may have no knowledge or experiments enough to answer your questions.

4.2 Questionnaire Survey

Beijing is the capital of China, and Tianjin is a coastal city. Assume that the real distance between them is not exactly known for us. It is more acceptable to regard such an unknown quantity as an uncertain variable than a random variable or a fuzzy variable. Chen and Ralescu [18] employed uncertain statistics to estimate the travel distance between Beijing and Tianjin. The consultation process is as follows:

Q1: May I ask you how far is from Beijing to Tianjin? What do you think is the minimum distance?

A1: 100km. *(an expert's experimental data* $(100, 0)$ *is acquired)*

Q2: What do you think is the maximum distance?

A2: 150km. *(an expert's experimental data* $(150, 1)$ *is acquired)*

Q3: What do you think is a likely distance?

A3: 130km.

Q4: What is the belief degree that the real distance is less than 130km?

A4: 0.6. *(an expert's experimental data* $(130, 0.6)$ *is acquired)*

Q5: Is there another number this distance may be?

A5: 140km.

Q6: What is the belief degree that the real distance is less than 140km?

A6: 0.9. *(an expert's experimental data* $(140, 0.9)$ *is acquired)*

Q7: Is there another number this distance may be?

A7: 120km.

Q8: What is the belief degree that the real distance is less than 120km?

A8: 0.3. *(an expert's experimental data* (120, 0.3) *is acquired)*

Q9: Is there another number this distance may be?

A9: No idea.

By using the questionnaire survey, five expert's experimental data of the travel distance between Beijing and Tianjin are acquired from the domain expert,

$$(100, 0),\ (120, 0.3),\ (130, 0.6),\ (140, 0.9),\ (150, 1). \tag{4.4}$$

4.3 Empirical Uncertainty Distribution

How do we determine the uncertainty distribution for an uncertain variable? Assume that we have obtained a set of expert's experimental data

$$(x_1, \alpha_1), (x_2, \alpha_2), \cdots, (x_n, \alpha_n) \tag{4.5}$$

that meet the following consistence condition (perhaps after a rearrangement)

$$x_1 < x_2 < \cdots < x_n, \quad 0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le 1. \tag{4.6}$$

Based on those expert's experimental data, Liu [129] suggested an empirical uncertainty distribution,

$$\Phi(x) = \begin{cases} 0, & \text{if } x < x_1 \\ \alpha_i + \dfrac{(\alpha_{i+1} - \alpha_i)(x - x_i)}{x_{i+1} - x_i}, & \text{if } x_i \le x \le x_{i+1},\ 1 \le i < n \\ 1, & \text{if } x > x_n. \end{cases} \tag{4.7}$$

Essentially, it is a type of linear interpolation method.

The empirical uncertainty distribution Φ determined by (4.7) has an expected value

$$E[\xi] = \frac{\alpha_1 + \alpha_2}{2} x_1 + \sum_{i=2}^{n-1} \frac{\alpha_{i+1} - \alpha_{i-1}}{2} x_i + \left(1 - \frac{\alpha_{n-1} + \alpha_n}{2}\right) x_n. \tag{4.8}$$

If all x_i's are nonnegative, then the k-th empirical moments are

$$E[\xi^k] = \alpha_1 x_1^k + \frac{1}{k+1} \sum_{i=1}^{n-1} \sum_{j=0}^{k} (\alpha_{i+1} - \alpha_i) x_i^j x_{i+1}^{k-j} + (1 - \alpha_n) x_n^k. \tag{4.9}$$

Example 4.1: Recall that the five expert's experimental data (100, 0), (120, 0.3), (130, 0.6), (140, 0.9), (150, 1) of the travel distance between Beijing and Tianjin have been acquired in Section 4.2. Based on those expert's experimental data, an empirical uncertainty distribution of travel distance is shown in Figure 4.3.

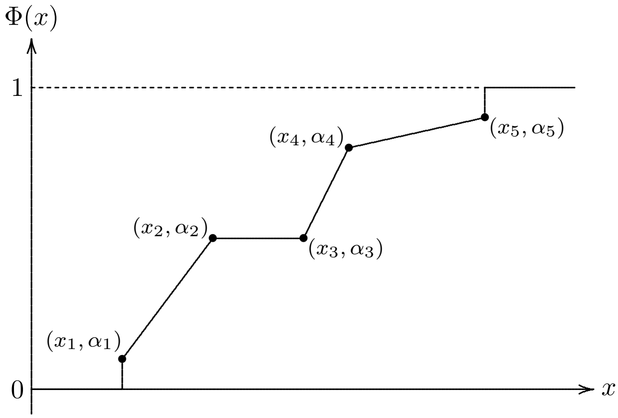

Figure 4.2: Empirical Uncertainty Distribution $\Phi(x)$. Reprinted from Liu [129].

4.4 Principle of Least Squares

Assume that an uncertainty distribution to be determined has a known functional form $\Phi(x|\theta)$ with an unknown parameter θ. In order to estimate the parameter θ, Liu [129] employed the principle of least squares that minimizes the sum of the squares of the distance of the expert's experimental data to the uncertainty distribution. This minimization can be performed in either the vertical or horizontal direction. If the expert's experimental data

$$(x_1,\alpha_1),(x_2,\alpha_2),\cdots,(x_n,\alpha_n) \tag{4.10}$$

are obtained and the vertical direction is accepted, then we have

$$\min_{\theta}\sum_{i=1}^{n}(\Phi(x_i|\theta)-\alpha_i)^2. \tag{4.11}$$

The optimal solution $\widehat{\theta}$ of (4.11) is called the least squares estimate of θ, and then the least squares uncertainty distribution is $\Phi(x|\widehat{\theta})$.

Example 4.2: Assume that an uncertainty distribution has a linear form with two unknown parameters a and b, i.e.,

$$\Phi(x)=\begin{cases} 0, & \text{if } x\le a\\ (x-a)/(b-a), & \text{if } a\le x\le b\\ 1, & \text{if } x\ge b. \end{cases} \tag{4.12}$$

We also assume the following expert's experimental data,

$$(1,0.15),\ (2,0.45),\ (3,0.55),\ (4,0.85),\ (5,0.95). \tag{4.13}$$

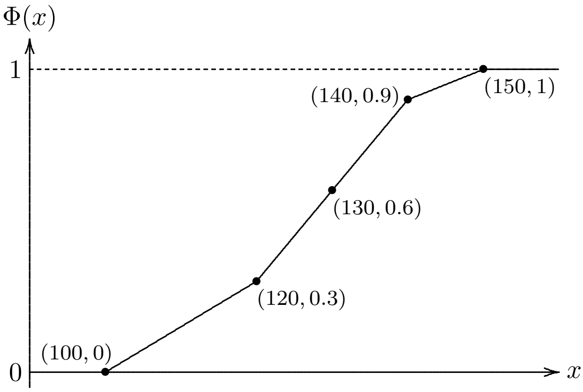

Figure 4.3: Empirical Uncertainty Distribution of Travel Distance between Beijing and Tianjin. Note that the empirical expected distance is 125.5km and the real distance is 127km in the google earth.

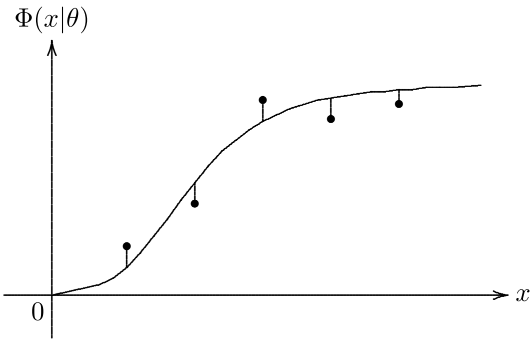

Figure 4.4: Principle of Least Squares

The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may yield that $a = 0.2273$, $b = 4.7727$ and the least squares uncertainty distribution is

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le 0.2273 \\ (x - 0.2273)/4.5454, & \text{if } 0.2273 \le x \le 4.7727 \\ 1, & \text{if } x \ge 4.7727. \end{cases} \tag{4.14}$$

Example 4.3: Assume that an uncertainty distribution has a lognormal form with two unknown parameters e and σ, i.e.,

$$\Phi(x|e,\sigma) = \left(1 + \exp\left(\frac{\pi(e - \ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}. \tag{4.15}$$

We also assume the following expert's experimental data,

$$(0.6, 0.1),\ (1.0, 0.3),\ (1.5, 0.4),\ (2.0, 0.6),\ (2.8, 0.8),\ (3.6, 0.9). \tag{4.16}$$

The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may yield that $e = 0.4825$, $\sigma = 0.7852$ and the least squares uncertainty distribution is

$$\Phi(x) = \left(1 + \exp\left(\frac{0.4825 - \ln x}{0.4329}\right)\right)^{-1}. \tag{4.17}$$

4.5 Method of Moments

Assume that a nonnegative uncertain variable has an uncertainty distribution

$$\Phi(x|\theta_1, \theta_2, \cdots, \theta_p) \tag{4.18}$$

with unknown parameters $\theta_1, \theta_2, \cdots, \theta_p$. Given a set of expert's experimental data

$$(x_1, \alpha_1), (x_2, \alpha_2), \cdots, (x_n, \alpha_n) \tag{4.19}$$

with

$$0 \le x_1 < x_2 < \cdots < x_n, \quad 0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \le 1, \tag{4.20}$$

Wang and Peng [230] proposed a method of moments to estimate the unknown parameters of uncertainty distribution. At first, the kth empirical moments of the expert's experimental data are defined as that of the corresponding empirical uncertainty distribution, i.e.,

$$\overline{\xi^k} = \alpha_1 x_1^k + \frac{1}{k+1}\sum_{i=1}^{n-1}\sum_{j=0}^{k}(\alpha_{i+1} - \alpha_i)x_i^j x_{i+1}^{k-j} + (1 - \alpha_n)x_n^k. \tag{4.21}$$

The moment estimates $\widehat{\theta}_1, \widehat{\theta}_2, \cdots, \widehat{\theta}_p$ are then obtained by equating the first p moments of $\Phi(x|\theta_1, \theta_2, \cdots, \theta_p)$ to the corresponding first p empirical moments. In other words, the moment estimates $\widehat{\theta}_1, \widehat{\theta}_2, \cdots, \widehat{\theta}_p$ should solve the system of equations,

$$\int_0^{+\infty}(1 - \Phi(\sqrt[k]{x} \,|\, \theta_1, \theta_2, \cdots, \theta_p))\mathrm{d}x = \overline{\xi^k}, \quad k = 1, 2, \cdots, p \tag{4.22}$$

where $\overline{\xi^1}, \overline{\xi^2}, \cdots, \overline{\xi^p}$ are empirical moments determined by (4.21).

Example 4.4: Assume that a questionnaire survey has successfully acquired the following expert's experimental data,

$$(1.2, 0.1),\ (1.5, 0.3),\ (1.8, 0.4),\ (2.5, 0.6),\ (3.9, 0.8),\ (4.6, 0.9). \tag{4.23}$$

Then the first three empirical moments are 2.5100, 7.7226 and 29.4936. We also assume that the uncertainty distribution to be determined has a zigzag form with three unknown parameters a, b and c, i.e.,

$$\Phi(x|a,b,c) = \begin{cases} 0, & \text{if } x \le a \\ (x-a)/2(b-a), & \text{if } a \le x \le b \\ (x+c-2b)/2(c-b), & \text{if } b \le x \le c \\ 1, & \text{if } x \ge c. \end{cases} \tag{4.24}$$

From the expert's experimental data, we may believe that the unknown parameters must be positive numbers. Thus the first three moments of the zigzag uncertainty distribution $\Phi(x|a,b,c)$ are

$$\frac{a+2b+c}{4},$$
$$\frac{a^2+ab+2b^2+bc+c^2}{6},$$
$$\frac{a^3+a^2b+ab^2+2b^3+b^2c+bc^2+c^3}{8}.$$

It follows from the method of moments that the unknown parameters a, b, c should solve the system of equations,

$$\begin{cases} a+2b+c = 4 \times 2.5100 \\ a^2+ab+2b^2+bc+c^2 = 6 \times 7.7226 \\ a^3+a^2b+ab^2+2b^3+b^2c+bc^2+c^3 = 8 \times 29.4936. \end{cases} \tag{4.25}$$

The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may yield that the moment estimates are $(a, b, c) = (0.9804, 2.0303, 4.9991)$ and the corresponding uncertainty distribution is

$$\Phi(x) = \begin{cases} 0, & \text{if } x \le 0.9804 \\ (x-0.9804)/2.0998, & \text{if } 0.9804 \le x \le 2.0303 \\ (x+0.9385)/5.9376, & \text{if } 2.0303 \le x \le 4.9991 \\ 1, & \text{if } x \ge 4.9991. \end{cases} \tag{4.26}$$

4.6 Multiple Domain Experts

Assume there are m domain experts and each produces an uncertainty distribution. Then we may get m uncertainty distributions $\Phi_1(x), \Phi_2(x), \cdots, \Phi_m(x)$. It was suggested by Liu [129] that the m uncertainty distributions should be aggregated to an uncertainty distribution

$$\Phi(x) = w_1\Phi_1(x) + w_2\Phi_2(x) + \cdots + w_m\Phi_m(x) \tag{4.27}$$

where $w_1, w_2, \cdots, w_m$ are convex combination coefficients (i.e., they are nonnegative numbers and $w_1 + w_2 + \cdots + w_n = 1$) representing weights of the domain experts. For example, we may set

$$w_i = \frac{1}{m}, \quad \forall i = 1, 2, \cdots, n. \tag{4.28}$$

Since $\Phi_1(x), \Phi_2(x), \cdots, \Phi_m(x)$ are uncertainty distributions, they are increasing functions taking values in $[0,1]$ and are not identical to either 0 or 1. It is easy to verify that their convex combination $\Phi(x)$ is also an increasing function taking values in $[0,1]$ and $\Phi(x) \not\equiv 0$, $\Phi(x) \not\equiv 1$. Hence $\Phi(x)$ is also an uncertainty distribution by Peng-Iwamura theorem.

4.7 Delphi Method

Delphi method was originally developed in the 1950s by the RAND Corporation based on the assumption that group experience is more valid than individual experience. This method asks the domain experts answer questionnaires in two or more rounds. After each round, a facilitator provides an anonymous summary of the answers from the previous round as well as the reasons that the domain experts provided for their opinions. Then the domain experts are encouraged to revise their earlier answers in light of the summary. It is believed that during this process the opinions of domain experts will converge to an appropriate answer. Wang, Gao and Guo [228] recast Delphi method as a process to determine uncertainty distributions. The main steps are listed as follows:

Step 1. The m domain experts provide their expert's experimental data,

$$(x_{ij}, \alpha_{ij}), \quad j = 1, 2, \cdots, n_i,\ i = 1, 2, \cdots, m. \tag{4.29}$$

Step 2. Use the i-th expert's experimental data $(x_{i1}, \alpha_{i1}), (x_{i2}, \alpha_{i2}), \cdots, (x_{in_i}, \alpha_{in_i})$ to generate the uncertainty distributions Φ_i of the i-th domain experts, $i = 1, 2, \cdots, m$, respectively.

Step 3. Compute $\Phi(x) = w_1\Phi_1(x) + w_2\Phi_2(x) + \cdots + w_m\Phi_m(x)$ where $w_1, w_2, \cdots, w_m$ are convex combination coefficients representing weights of the domain experts.

Step 4. If $|\alpha_{ij} - \Phi(x_{ij})|$ are less than a given level $\varepsilon > 0$ for all i and j, then go to Step 5. Otherwise, the i-th domain experts receive the summary (for example, the function Φ obtained in the previous round and the reasons of other experts), and then provide a set of revised expert's experimental data $(x_{i1}, \alpha_{i1}), (x_{i2}, \alpha_{i2}), \cdots, (x_{in_i}, \alpha_{in_i})$ for $i = 1, 2, \cdots, m$. Go to Step 2.

Step 5. The last function Φ is the uncertainty distribution to be determined.

4.8 Bibliographic Notes

The study of uncertain statistics was started by Liu [129] in 2010 in which a questionnaire survey for collecting expert's experimental data was designed. It was shown among others by Chen and Ralescu [18] that the questionnaire survey may successfully acquire the expert's experimental data.

Parametric uncertain statistics assumes that the uncertainty distribution to be determined has a known functional form but with unknown parameters. In order to estimate the unknown parameters, Liu [129] suggested the principle of least squares, and Wang and Peng [230] proposed the method of moments.

Nonparametric uncertain statistics does not rely on the expert's experimental data belonging to any particular uncertainty distribution. In order to determine the uncertainty distributions, Liu [129] introduced the linear interpolation method (i.e., empirical uncertainty distribution), and Chen and Ralescu [18] proposed a series of spline interpolation methods.

When multiple domain experts are available, Wang, Gao and Guo [228] recast Delphi method as a process to determine uncertainty distributions.

Chapter 5

Uncertain Risk Analysis

The term *risk* has been used in different ways in literature. Here the risk is defined as the “accidental loss” plus “uncertain measure of such loss”. Uncertain risk analysis is a tool to quantify risk via uncertainty theory. One main feature of this topic is to model events that almost never occur. This chapter will introduce a definition of risk index and provide some useful formulas for calculating risk index. This chapter will also discuss structural risk analysis and investment risk analysis in uncertain environments.

5.1 Loss Function

A system usually contains some factors $\xi_1, \xi_2, \cdots, \xi_n$ that may be understood as lifetime, strength, demand, production rate, cost, profit, and resource. Generally speaking, some specified loss is dependent on those factors. Although loss is a problem-dependent concept, usually such a loss may be represented by a loss function.

Definition 5.1 *Consider a system with factors $\xi_1, \xi_2, \cdots, \xi_n$. A function f is called a loss function if some specified loss occurs if and only if*

$$f(\xi_1, \xi_2, \cdots, \xi_n) > 0. \tag{5.1}$$

Example 5.1: Consider a series system in which there are n elements whose lifetimes are uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$. Such a system works whenever all elements work. Thus the system lifetime is

$$\xi = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n. \tag{5.2}$$

If the loss is understood as the case that the system fails before the time T, then we have a loss function

$$f(\xi_1, \xi_2, \cdots, \xi_n) = T - \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n. \tag{5.3}$$

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5_6

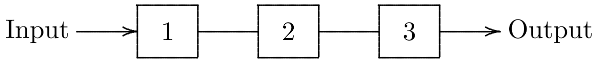

Figure 5.1: A Series System. Reprinted from Liu [129].

Hence the system fails if and only if $f(\xi_1,\xi_2,\cdots,\xi_n)>0$.

Example 5.2: Consider a parallel system in which there are n elements whose lifetimes are uncertain variables $\xi_1,\xi_2,\cdots,\xi_n$. Such a system works whenever at least one element works. Thus the system lifetime is

$$\xi=\xi_1\vee\xi_2\vee\cdots\vee\xi_n. \tag{5.4}$$

If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f(\xi_1,\xi_2,\cdots,\xi_n)=T-\xi_1\vee\xi_2\vee\cdots\vee\xi_n. \tag{5.5}$$

Hence the system fails if and only if $f(\xi_1,\xi_2,\cdots,\xi_n)>0$.

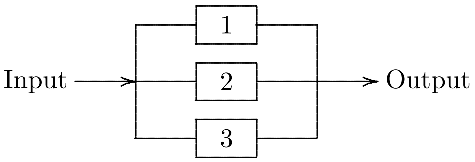

Figure 5.2: A Parallel System. Reprinted from Liu [129].

Example 5.3: Consider a k-out-of-n system in which there are n elements whose lifetimes are uncertain variables $\xi_1,\xi_2,\cdots,\xi_n$. Such a system works whenever at least k of n elements work. Thus the system lifetime is

$$\xi=k\text{-max}\,[\xi_1,\xi_2,\cdots,\xi_n]. \tag{5.6}$$

If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f(\xi_1,\xi_2,\cdots,\xi_n)=T-k\text{-max}\,[\xi_1,\xi_2,\cdots,\xi_n]. \tag{5.7}$$

Hence the system fails if and only if $f(\xi_1,\xi_2,\cdots,\xi_n)>0$. Note that a series system is an n-out-of-n system, and a parallel system is a 1-out-of-n system.

Example 5.4: Consider a standby system in which there are n redundant elements whose lifetimes are $\xi_1,\xi_2,\cdots,\xi_n$. For this system, only one element is active, and one of the redundant elements begins to work only when the active element fails. Thus the system lifetime is

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n. \tag{5.8}$$

If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f(\xi_1, \xi_2, \cdots, \xi_n) = T - (\xi_1 + \xi_2 + \cdots + \xi_n). \tag{5.9}$$

Hence the system fails if and only if $f(\xi_1, \xi_2, \cdots, \xi_n) > 0$.

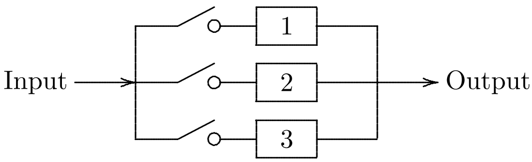

Figure 5.3: A Standby System

5.2 Risk Index

In practice, the factors $\xi_1, \xi_2, \cdots, \xi_n$ of a system are usually uncertain variables rather than known constants. Thus the risk index is defined as the uncertain measure that some specified loss occurs.

Definition 5.2 *(Liu [128]) Assume that a system contains uncertain factors $\xi_1, \xi_2, \cdots, \xi_n$ and has a loss function f. Then the risk index is the uncertain measure that the system is loss-positive, i.e.,*

$$Risk = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) > 0\}. \tag{5.10}$$

Theorem 5.1 *(Liu [128], Risk Index Theorem) Assume a system contains independent uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the loss function $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then the risk index is just the root α of the equation*

$$f(\Phi_1^{-1}(1-\alpha), \cdots, \Phi_m^{-1}(1-\alpha), \Phi_{m+1}^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)) = 0. \tag{5.11}$$

Proof: It follows from Definition 5.2 and Theorem 2.21 immediately.

Remark 5.1: Since $f(\Phi_1^{-1}(1-\alpha), \cdots, \Phi_m^{-1}(1-\alpha), \Phi_{m+1}^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha))$ is a strictly decreasing function with respect to α, its root α may be estimated by the bisection method.

Remark 5.2: Keep in mind that sometimes the equation (5.11) may not have a root. In this case, if

$$f(\Phi_1^{-1}(1-\alpha),\cdots,\Phi_m^{-1}(1-\alpha),\Phi_{m+1}^{-1}(\alpha),\cdots,\Phi_n^{-1}(\alpha))<0 \tag{5.12}$$

for all α, then we set the root $\alpha=0$; and if

$$f(\Phi_1^{-1}(1-\alpha),\cdots,\Phi_m^{-1}(1-\alpha),\Phi_{m+1}^{-1}(\alpha),\cdots,\Phi_n^{-1}(\alpha))>0 \tag{5.13}$$

for all α, then we set the root $\alpha=1$.

5.3 Series System

Consider a series system in which there are n elements whose lifetimes are independent uncertain variables $\xi_1,\xi_2,\cdots,\xi_n$ with uncertainty distributions $\Phi_1,\Phi_2,\cdots,\Phi_n$, respectively. If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f(\xi_1,\xi_2,\cdots,\xi_n)=T-\xi_1\wedge\xi_2\wedge\cdots\wedge\xi_n \tag{5.14}$$

and the risk index is

$$Risk=\mathcal{M}\{f(\xi_1,\xi_2,\cdots,\xi_n)>0\}. \tag{5.15}$$

Since f is a strictly decreasing function with respect to $\xi_1,\xi_2,\cdots,\xi_n$, the risk index theorem says that the risk index is just the root α of the equation

$$\Phi_1^{-1}(\alpha)\wedge\Phi_2^{-1}(\alpha)\wedge\cdots\wedge\Phi_n^{-1}(\alpha)=T. \tag{5.16}$$

It is easy to verify that

$$Risk=\Phi_1(T)\vee\Phi_2(T)\vee\cdots\vee\Phi_n(T). \tag{5.17}$$

5.4 Parallel System

Consider a parallel system in which there are n elements whose lifetimes are independent uncertain variables $\xi_1,\xi_2,\cdots,\xi_n$ with uncertainty distributions $\Phi_1,\Phi_2,\cdots,\Phi_n$, respectively. If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f(\xi_1,\xi_2,\cdots,\xi_n)=T-\xi_1\vee\xi_2\vee\cdots\vee\xi_n \tag{5.18}$$

and the risk index is

$$Risk=\mathcal{M}\{f(\xi_1,\xi_2,\cdots,\xi_n)>0\}. \tag{5.19}$$

Since f is a strictly decreasing function with respect to $\xi_1,\xi_2,\cdots,\xi_n$, the risk index theorem says that the risk index is just the root α of the equation

$$\Phi_1^{-1}(\alpha)\vee\Phi_2^{-1}(\alpha)\vee\cdots\vee\Phi_n^{-1}(\alpha)=T. \tag{5.20}$$

It is easy to verify that

$$Risk=\Phi_1(T)\wedge\Phi_2(T)\wedge\cdots\wedge\Phi_n(T). \tag{5.21}$$

5.5 k-out-of-n System

Consider a k-out-of-n system in which there are n elements whose lifetimes are independent uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f(\xi_1, \xi_2, \cdots, \xi_n) = T - k\text{-max}\,[\xi_1, \xi_2, \cdots, \xi_n] \tag{5.22}$$

and the risk index is

$$Risk = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) > 0\}. \tag{5.23}$$

Since f is a strictly decreasing function with respect to $\xi_1, \xi_2, \cdots, \xi_n$, the risk index theorem says that the risk index is just the root α of the equation

$$k\text{-max}\,[\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)] = T. \tag{5.24}$$

It is easy to verify that

$$Risk = k\text{-min}\,[\Phi_1(T), \Phi_2(T), \cdots, \Phi_n(T)]. \tag{5.25}$$

Note that a series system is essentially an n-out-of-n system. In this case, the risk index formula (5.25) becomes (5.17). In addition, a parallel system is essentially a 1-out-of-n system. In this case, the risk index formula (5.25) becomes (5.21).

5.6 Standby System

Consider a standby system in which there are n elements whose lifetimes are independent uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f(\xi_1, \xi_2, \cdots, \xi_n) = T - (\xi_1 + \xi_2 + \cdots + \xi_n) \tag{5.26}$$

and the risk index is

$$Risk = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) > 0\}. \tag{5.27}$$

Since f is a strictly decreasing function with respect to $\xi_1, \xi_2, \cdots, \xi_n$, the risk index theorem says that the risk index is just the root α of the equation

$$\Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha) = T. \tag{5.28}$$

5.7 Structural Risk Analysis

Consider a structural system in which the strengths and loads are assumed to be uncertain variables. We will suppose that a structural system fails whenever for each rod, the load variable exceeds its strength variable. If the structural risk index is defined as the uncertain measure that the structural system fails, then

$$Risk = \mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i < \eta_i)\right\} \tag{5.29}$$

where $\xi_1, \xi_2, \cdots, \xi_n$ are strength variables, and $\eta_1, \eta_2, \cdots, \eta_n$ are load variables of the n rods.

Example 5.5: (The Simplest Case) Assume there is only a single strength variable ξ and a single load variable η with continuous uncertainty distributions Φ and Ψ, respectively. In this case, the structural risk index is

$$Risk = \mathcal{M}\{\xi < \eta\}.$$

It follows from the risk index theorem that the risk index is just the root α of the equation

$$\Phi^{-1}(\alpha) = \Psi^{-1}(1-\alpha). \tag{5.30}$$

Especially, if the strength variable ξ has a normal uncertainty distribution $\mathcal{N}(e_s, \sigma_s)$ and the load variable η has a normal uncertainty distribution $\mathcal{N}(e_l, \sigma_l)$, then the structural risk index is

$$Risk = \left(1 + \exp\left(\frac{\pi(e_s - e_l)}{\sqrt{3}(\sigma_s + \sigma_l)}\right)\right)^{-1}. \tag{5.31}$$

Example 5.6: (Constant Loads) Assume the uncertain strength variables $\xi_1, \xi_2, \cdots, \xi_n$ are independent and have continuous uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. In many cases, the load variables $\eta_1, \eta_2, \cdots, \eta_n$ degenerate to crisp values $c_1, c_2, \cdots, c_n$ (for example, weight limits allowed by the legislation), respectively. In this case, it follows from (5.29) and independence that the structural risk index is

$$Risk = \mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i < c_i)\right\} = \bigvee_{i=1}^{n}\mathcal{M}\{\xi_i < c_i\}.$$

That is,

$$Risk = \Phi_1(c_1) \vee \Phi_2(c_2) \vee \cdots \vee \Phi_n(c_n). \tag{5.32}$$

Example 5.7: (Independent Load Variables) Assume the uncertain strength variables $\xi_1, \xi_2, \cdots, \xi_n$ are independent and have continuous uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Also assume the uncertain load variables $\eta_1, \eta_2, \cdots, \eta_n$ are independent and have continuous uncertainty distributions $\Psi_1, \Psi_2, \cdots, \Psi_n$, respectively. In this case, it follows from (5.29) and independence that the structural risk index is

$$Risk = \mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i < \eta_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i < \eta_i\}.$$

That is,

$$Risk = \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n \tag{5.33}$$

where α_i are the roots of the equations

$$\Phi_i^{-1}(\alpha) = \Psi_i^{-1}(1-\alpha) \tag{5.34}$$

for $i = 1, 2, \cdots, n$, respectively.

However, generally speaking, the load variables $\eta_1, \eta_2, \cdots, \eta_n$ are neither constants nor independent. For examples, the load variables $\eta_1, \eta_2, \cdots, \eta_n$ may be functions of independent uncertain variables $\tau_1, \tau_2, \cdots, \tau_m$. In this case, the formula (5.33) is no longer valid. Thus we have to deal with those structural systems case by case.

Example 5.8: (Series System) Consider a structural system shown in Figure 5.4 that consists of n rods in series and an object. Assume that the strength variables of the n rods are uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. We also assume that the gravity of the object is an uncertain variable η with uncertainty distribution Ψ. For each i $(1 \le i \le n)$, the load variable of the rod i is just the gravity η of the object. Thus the structural system fails whenever the load variable η exceeds at least one of the strength variables $\xi_1, \xi_2, \cdots, \xi_n$. Hence the structural risk index is

$$Risk = \mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i < \eta)\right\} = \mathcal{M}\{\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n < \eta\}.$$

Define the loss function as

$$f(\xi_1, \xi_2, \cdots, \xi_n, \eta) = \eta - \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n.$$

Then

$$Risk = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n, \eta) > 0\}.$$

Since the loss function f is strictly increasing with respect to η and strictly decreasing with respect to $\xi_1, \xi_2, \cdots, \xi_n$, it follows from the risk index theorem that the risk index is just the root α of the equation

$$\Psi^{-1}(1-\alpha) - \Phi_1^{-1}(\alpha) \wedge \Phi_2^{-1}(\alpha) \wedge \cdots \wedge \Phi_n^{-1}(\alpha) = 0. \tag{5.35}$$

Or equivalently, let α_i be the roots of the equations

$$\Psi^{-1}(1-\alpha) = \Phi_i^{-1}(\alpha) \tag{5.36}$$

for $i = 1, 2, \cdots, n$, respectively. Then the structural risk index is

$$Risk = \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n. \tag{5.37}$$

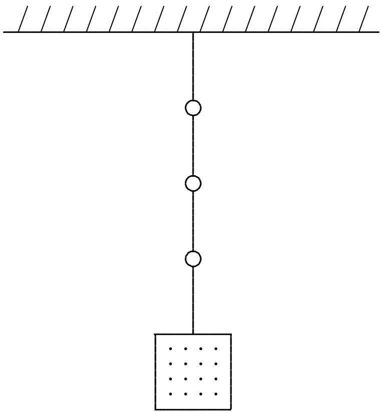

Figure 5.4: A Structural System with n Rods and an Object

Example 5.9: Consider a structural system shown in Figure 5.5 that consists of 2 rods and an object. Assume that the strength variables of the left and right rods are uncertain variables ξ_1 and ξ_2 with uncertainty distributions Φ_1 and Φ_2, respectively. We also assume that the gravity of the object is an uncertain variable η with uncertainty distribution Ψ. In this case, the load variables of left and right rods are respectively equal to

$$\frac{\eta \sin\theta_2}{\sin(\theta_1+\theta_2)}, \quad \frac{\eta \sin\theta_1}{\sin(\theta_1+\theta_2)}.$$

Thus the structural system fails whenever for any one rod, the load variable exceeds its strength variable. Hence the structural risk index is

$$\begin{aligned} Risk &= \mathcal{M}\left\{\left(\xi_1 < \frac{\eta \sin\theta_2}{\sin(\theta_1+\theta_2)}\right) \cup \left(\xi_2 < \frac{\eta \sin\theta_1}{\sin(\theta_1+\theta_2)}\right)\right\} \\ &= \mathcal{M}\left\{\left(\frac{\xi_1}{\sin\theta_2} < \frac{\eta}{\sin(\theta_1+\theta_2)}\right) \cup \left(\frac{\xi_2}{\sin\theta_1} < \frac{\eta}{\sin(\theta_1+\theta_2)}\right)\right\} \\ &= \mathcal{M}\left\{\frac{\xi_1}{\sin\theta_2} \wedge \frac{\xi_2}{\sin\theta_1} < \frac{\eta}{\sin(\theta_1+\theta_2)}\right\} \end{aligned}$$

Define the loss function as

$$f(\xi_1, \xi_2, \eta) = \frac{\eta}{\sin(\theta_1 + \theta_2)} - \frac{\xi_1}{\sin\theta_2} \wedge \frac{\xi_2}{\sin\theta_1}.$$

Then

$$Risk = \mathcal{M}\{f(\xi_1, \xi_2, \eta) > 0\}.$$

Since the loss function f is strictly increasing with respect to η and strictly decreasing with respect to ξ_1, ξ_2, it follows from the risk index theorem that the risk index is just the root α of the equation

$$\frac{\Psi^{-1}(1-\alpha)}{\sin(\theta_1 + \theta_2)} - \frac{\Phi_1^{-1}(\alpha)}{\sin\theta_2} \wedge \frac{\Phi_2^{-1}(\alpha)}{\sin\theta_1} = 0. \tag{5.38}$$

Or equivalently, let α_1 be the root of the equation

$$\frac{\Psi^{-1}(1-\alpha)}{\sin(\theta_1 + \theta_2)} = \frac{\Phi_1^{-1}(\alpha)}{\sin\theta_2} \tag{5.39}$$

and let α_2 be the root of the equation

$$\frac{\Psi^{-1}(1-\alpha)}{\sin(\theta_1 + \theta_2)} = \frac{\Phi_2^{-1}(\alpha)}{\sin\theta_1}. \tag{5.40}$$

Then the structural risk index is

$$Risk = \alpha_1 \vee \alpha_2. \tag{5.41}$$

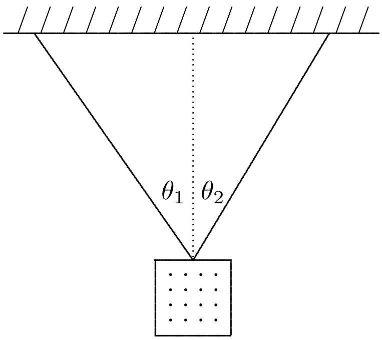

Figure 5.5: A Structural System with 2 Rods and an Object

5.8 Investment Risk Analysis

Assume that an investor has n projects whose returns are uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$. If the loss is understood as the case that total return $\xi_1 + \xi_2 +$

$\cdots + \xi_n$ is below a predetermined value c (e.g., the interest rate), then the investment risk index is

$$Risk = \mathcal{M}\{\xi_1 + \xi_2 + \cdots + \xi_n < c\}. \tag{5.42}$$

If $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, then the investment risk index is just the root α of the equation

$$\Phi_1^{-1}(\alpha) + \Phi_2^{-1}(\alpha) + \cdots + \Phi_n^{-1}(\alpha) = c. \tag{5.43}$$

5.9 Value-at-Risk

As a substitute of risk index (5.10), a concept of value-at-risk is given by the following definition.

Definition 5.3 *(Peng [183]) Assume that a system contains uncertain factors $\xi_1, \xi_2, \cdots, \xi_n$ and has a loss function f. Then the value-at-risk is defined as*

$$\text{VaR}(\alpha) = \sup\{x \mid \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) \geq x\} \geq \alpha\}. \tag{5.44}$$

Note that $\text{VaR}(\alpha)$ represents the maximum possible loss when α percent of the right tail distribution is ignored. In other words, the loss $f(\xi_1, \xi_2, \cdots, \xi_n)$ will exceed $\text{VaR}(\alpha)$ with uncertain measure α. See Figure 5.6. If $\Phi(x)$ is the uncertainty distribution of $f(\xi_1, \xi_2, \cdots, \xi_n)$, then

$$\text{VaR}(\alpha) = \sup\{x \mid \Phi(x) \leq 1 - \alpha\}. \tag{5.45}$$

If its inverse uncertainty distribution $\Phi^{-1}(\alpha)$ exists, then

$$\text{VaR}(\alpha) = \Phi^{-1}(1 - \alpha). \tag{5.46}$$

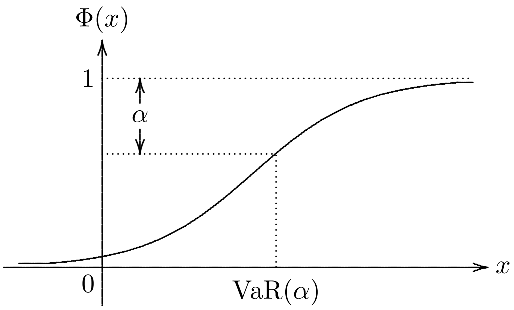

Figure 5.6: Value-at-Risk

Theorem 5.2 *(Peng [183]) The value-at-risk* VaR(α) *is a monotone decreasing function with respect to* α.

Proof: Let α_1 and α_2 be two numbers with $0 < \alpha_1 < \alpha_2 \le 1$. Then for any number $r < \text{VaR}(\alpha_2)$, we have

$$\mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) \ge r\} \ge \alpha_2 > \alpha_1.$$

Thus, by the definition of value-at-risk, we obtain $\text{VaR}(\alpha_1) \le r < \text{VaR}(\alpha_2)$. That is, VaR($\alpha$) is a monotone decreasing function with respect to α.

Theorem 5.3 *(Peng [183]) Assume a system contains independent uncertain variables* $\xi_1, \xi_2, \cdots, \xi_n$ *with regular uncertainty distributions* $\Phi_1, \Phi_2, \cdots, \Phi_n$*, respectively. If the loss function* $f(\xi_1, \xi_2, \cdots, \xi_n)$ *is strictly increasing with respect to* $\xi_1, \xi_2, \cdots, \xi_m$ *and strictly decreasing with respect to* $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$*, then*

$$\text{VaR}(\alpha) = f(\Phi_1^{-1}(1-\alpha), \cdots, \Phi_m^{-1}(1-\alpha), \Phi_{m+1}^{-1}(\alpha), \cdots, \Phi_n^{-1}(\alpha)). \tag{5.47}$$

Proof: It follows from the operational law of uncertain variables that the loss $f(\xi_1, \xi_2, \cdots, \xi_n)$ has an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

The theorem follows from (5.46) immediately.

5.10 Expected Loss

Liu and Ralescu [151] proposed a concept of expected loss that is the expected value of the loss $f(\xi_1, \xi_2, \cdots, \xi_n)$ given $f(\xi_1, \xi_2, \cdots, \xi_n) > 0$. A formal definition is given below.

Definition 5.4 *(Liu and Ralescu [151]) Assume that a system contains uncertain factors* $\xi_1, \xi_2, \cdots, \xi_n$ *and has a loss function* f*. Then the expected loss is defined as*

$$L = \int_0^{+\infty} \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) \ge x\}\mathrm{d}x. \tag{5.48}$$

If $\Phi(x)$ is the uncertainty distribution of the loss $f(\xi_1, \xi_2, \cdots, \xi_n)$, then we immediately have

$$L = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x. \tag{5.49}$$

If its inverse uncertainty distribution $\Phi^{-1}(\alpha)$ exists, then the expected loss is

$$L = \int_0^1 \left(\Phi^{-1}(\alpha)\right)^+ \mathrm{d}\alpha. \tag{5.50}$$

Theorem 5.4 *(Liu and Ralescu [154]) Assume a system contains independent uncertain variables $\xi_1, \xi_2, \cdots, \xi_n$ with regular uncertainty distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. If the loss function $f(\xi_1, \xi_2, \cdots, \xi_n)$ is strictly increasing with respect to $\xi_1, \xi_2, \cdots, \xi_m$ and strictly decreasing with respect to $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$, then the expected loss is*

$$L = \int_0^1 f^+(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha. \quad (5.51)$$

Proof: It follows from the operational law of uncertain variables that the loss $f(\xi_1, \xi_2, \cdots, \xi_n)$ has an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \cdots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \cdots, \Phi_n^{-1}(1-\alpha)).$$

The theorem follows from (5.50) immediately.

5.11 Hazard Distribution

Suppose that ξ is the lifetime of some element. Here we assume it is an uncertain variable with a prior uncertainty distribution Φ. At some time t, it is observed that the element is working. What is the residual lifetime of the element? The following definition answers this question.

Definition 5.5 *(Liu [128]) Let ξ be a nonnegative uncertain variable representing lifetime of some element. If ξ has a prior uncertainty distribution Φ, then the hazard distribution at time t is*

$$\Phi(x|t) = \begin{cases} 0, & \text{if } \Phi(x) \le \Phi(t) \\ \dfrac{\Phi(x)}{1-\Phi(t)} \wedge 0.5, & \text{if } \Phi(t) < \Phi(x) \le (1+\Phi(t))/2 \\ \dfrac{\Phi(x)-\Phi(t)}{1-\Phi(t)}, & \text{if } (1+\Phi(t))/2 \le \Phi(x) \end{cases} \quad (5.52)$$

that is just the conditional uncertainty distribution of ξ given $\xi > t$.

The hazard distribution is essentially the posterior uncertainty distribution just after time t given that it is working at time t.

Exercise 5.1: Let ξ be a linear uncertain variable $\mathcal{L}(a, b)$, and t a real number with $a < t < b$. Show that the hazard distribution at time t is

$$\Phi(x|t) = \begin{cases} 0, & \text{if } x \le t \\ \dfrac{x-a}{b-t} \wedge 0.5, & \text{if } t < x \le (b+t)/2 \\ \dfrac{x-t}{b-t} \wedge 1, & \text{if } (b+t)/2 \le x. \end{cases}$$

Theorem 5.5 *(Liu [128], Conditional Risk Index Theorem) Assume that a system contains uncertain factors* $\xi_1, \xi_2, \cdots, \xi_n$*, and has a loss function* f*. Suppose* $\xi_1, \xi_2, \cdots, \xi_n$ *are independent uncertain variables with uncertainty distributions* $\Phi_1, \Phi_2, \cdots, \Phi_n$*, respectively, and* $f(\xi_1, \xi_2, \cdots, \xi_n)$ *is strictly increasing with respect to* $\xi_1, \xi_2, \cdots, \xi_m$ *and strictly decreasing with respect to* $\xi_{m+1}, \xi_{m+2}, \cdots, \xi_n$*. If it is observed that all elements are working at some time* t*, then the risk index is just the root* α *of the equation*

$$f(\Phi_1^{-1}(1-\alpha|t), \cdots, \Phi_m^{-1}(1-\alpha|t), \Phi_{m+1}^{-1}(\alpha|t), \cdots, \Phi_n^{-1}(\alpha|t)) = 0 \tag{5.53}$$

where $\Phi_i(x|t)$ *are hazard distributions determined by*

$$\Phi_i(x|t) = \begin{cases} 0, & \text{if } \Phi_i(x) \le \Phi_i(t) \\ \dfrac{\Phi_i(x)}{1-\Phi_i(t)} \wedge 0.5, & \text{if } \Phi_i(t) < \Phi_i(x) \le (1+\Phi_i(t))/2 \\ \dfrac{\Phi_i(x)-\Phi_i(t)}{1-\Phi_i(t)}, & \text{if } (1+\Phi_i(t))/2 \le \Phi_i(x) \end{cases} \tag{5.54}$$

for $i = 1, 2, \cdots, n$*.*

Proof: It follows from Definition 5.5 that each hazard distribution of element is determined by (5.54). Thus the conditional risk index is obtained by Theorem 5.1 immediately.

5.12 Bibliographic Notes

Uncertain risk analysis was proposed by Liu [128] in 2010 in which a risk index was defined and a risk index theorem was proved. This tool was also successfully applied among others to structural risk analysis and investment risk analysis.

As a substitute of risk index, Peng [183] suggested a concept of value-at-risk that is the maximum possible loss when the right tail distribution is ignored. In addition, Liu and Ralescu [151, 154] investigated the concept of expected loss that takes into account not only the chance of the loss but also its severity.

Chapter 6

Uncertain Reliability Analysis

Uncertain reliability analysis is a tool to deal with system reliability via uncertainty theory. This chapter will introduce a definition of reliability index and provide some useful formulas for calculating reliability index.

6.1 Structure Function

Many real systems may be simplified to a Boolean system in which each element (including the system itself) has two states: working and failure. Let Boolean variables x_i denote the states of elements i for $i = 1, 2, \cdots, n$, and

$$x_i = \begin{cases} 1, & \text{if element } i \text{ works} \\ 0, & \text{if element } i \text{ fails.} \end{cases} \tag{6.1}$$

We also suppose the Boolean variable X indicates the state of the system, i.e.,

$$X = \begin{cases} 1, & \text{if the system works} \\ 0, & \text{if the system fails.} \end{cases} \tag{6.2}$$

Usually, the state of the system is completely determined by the states of its elements via the so-called structure function.

Definition 6.1 *Assume that X is a Boolean system containing elements $x_1, x_2, \cdots, x_n$. A Boolean function f is called a structure function of X if*

$$X = 1 \textit{ if and only if } f(x_1, x_2, \cdots, x_n) = 1. \tag{6.3}$$

It is obvious that $X = 0$ if and only if $f(x_1, x_2, \cdots, x_n) = 0$ whenever f is indeed the structure function of the system.

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DOI 10.1007/978-3-662-44354-5_7

Example 6.1: For a series system, the structure function is a mapping from $\{0,1\}^n$ to $\{0,1\}$, i.e.,

$$f(x_1, x_2, \cdots, x_n) = x_1 \wedge x_2 \wedge \cdots \wedge x_n. \tag{6.4}$$

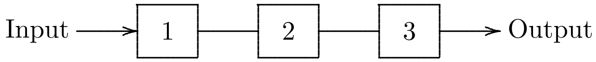


Figure 6.1: A Series System. Reprinted from Liu [129].

Example 6.2: For a parallel system, the structure function is a mapping from $\{0,1\}^n$ to $\{0,1\}$, i.e.,

$$f(x_1, x_2, \cdots, x_n) = x_1 \vee x_2 \vee \cdots \vee x_n. \tag{6.5}$$

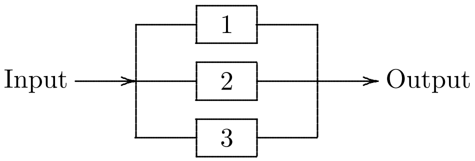


Figure 6.2: A Parallel System. Reprinted from Liu [129].

Example 6.3: For a k-out-of-n system that works whenever at least k of the n elements work, the structure function is a mapping from $\{0,1\}^n$ to $\{0,1\}$, i.e.,

$$f(x_1, x_2, \cdots, x_n) = \begin{cases} 1, & \text{if } x_1 + x_2 + \cdots + x_n \ge k \\ 0, & \text{if } x_1 + x_2 + \cdots + x_n < k. \end{cases} \tag{6.6}$$

Especially, when $k = 1$, it is a parallel system; when $k = n$, it is a series system.

6.2 Reliability Index

The element in a Boolean system is usually represented by a Boolean uncertain variable, i.e.,

$$\xi = \begin{cases} 1 \text{ with uncertain measure } a \\ 0 \text{ with uncertain measure } 1 - a. \end{cases} \tag{6.7}$$

In this case, we will say ξ is an uncertain element with reliability a. Reliability index is defined as the uncertain measure that the system is working.

Definition 6.2 *(Liu [128]) Assume a Boolean system has uncertain elements* $\xi_1, \xi_2, \cdots, \xi_n$ *and a structure function* f*. Then the reliability index is the uncertain measure that the system is working, i.e.,*

$$\text{Reliability} = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) = 1\}. \tag{6.8}$$

Theorem 6.1 *(Liu [128], Reliability Index Theorem) Assume that a system contains uncertain elements* $\xi_1, \xi_2, \cdots, \xi_n$*, and has a structure function* f*. If* $\xi_1, \xi_2, \cdots, \xi_n$ *are independent uncertain elements with reliabilities* $a_1, a_2, \cdots$*,* a_n*, respectively, then the reliability index is*

$$\text{Reliability} = \begin{cases} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \text{if} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \sup\limits_{f(x_1,x_2,\cdots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \text{if} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) \ge 0.5 \end{cases} \tag{6.9}$$

where x_i *take values either 0 or 1, and* ν_i *are defined by*

$$\nu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \tag{6.10}$$

for $i = 1, 2, \cdots, n$*, respectively.*

Proof: Since $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean uncertain variables and f is a Boolean function, the equation (6.9) follows from Definition 6.2 and Theorem 2.23 immediately.

6.3 Series System

Consider a series system having independent uncertain elements $\xi_1, \xi_2, \cdots, \xi_n$ with reliabilities $a_1, a_2, \cdots, a_n$, respectively. Note that the structure function is

$$f(x_1, x_2, \cdots, x_n) = x_1 \wedge x_2 \wedge \cdots \wedge x_n. \tag{6.11}$$

It follows from the reliability index theorem that the reliability index is

$$\text{Reliability} = \mathcal{M}\{\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n = 1\} = a_1 \wedge a_2 \wedge \cdots \wedge a_n. \tag{6.12}$$

6.4 Parallel System

Consider a parallel system having independent uncertain elements $\xi_1, \xi_2, \cdots,$ ξ_n with reliabilities $a_1, a_2, \cdots, a_n$, respectively. Note that the structure function is

$$f(x_1, x_2, \cdots, x_n) = x_1 \vee x_2 \vee \cdots \vee x_n. \tag{6.13}$$

It follows from the reliability index theorem that the reliability index is

$$Reliability = \mathcal{M}\{\xi_1 \vee \xi_2 \vee \cdots \vee \xi_n = 1\} = a_1 \vee a_2 \vee \cdots \vee a_n. \tag{6.14}$$

6.5 k-out-of-n System

Consider a k-out-of-n system having independent uncertain elements $\xi_1, \xi_2, \cdots, \xi_n$ with reliabilities $a_1, a_2, \cdots, a_n$, respectively. Note that the structure function has a Boolean form,

$$f(x_1, x_2, \cdots, x_n) = \begin{cases} 1, & \text{if } x_1 + x_2 + \cdots + x_n \geq k \\ 0, & \text{if } x_1 + x_2 + \cdots + x_n < k. \end{cases} \tag{6.15}$$

It follows from the reliability index theorem that the reliability index is the kth largest value of $a_1, a_2, \cdots, a_n$, i.e.,

$$Reliability = k\text{-max}[a_1, a_2, \cdots, a_n]. \tag{6.16}$$

Note that a series system is essentially an n-out-of-n system. In this case, the reliability index formula (6.16) becomes (6.12). In addition, a parallel system is essentially a 1-out-of-n system. In this case, the reliability index formula (6.16) becomes (6.14).

6.6 General System

It is almost impossible to find an analytic formula of reliability risk for general systems. In this case, we have to employ numerical method.

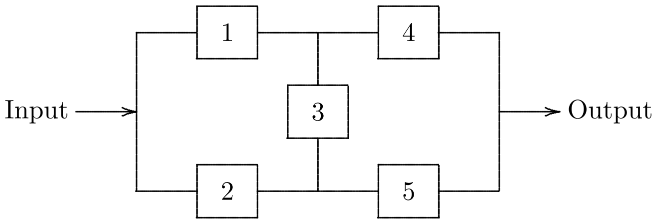


Figure 6.3: A Bridge System. Reprinted from Liu [129].

Consider a bridge system shown in Figure 6.3 that consists of 5 independent uncertain elements whose states are denoted by $\xi_1, \xi_2, \xi_3, \xi_4, \xi_5$. Assume each path works if and only if all elements on which are working and the system works if and only if there is a path of working elements. Then the structure function of the bridge system is

$$f(x_1, x_2, x_3, x_4, x_5) = (x_1 \wedge x_4) \vee (x_2 \wedge x_5) \vee (x_1 \wedge x_3 \wedge x_5) \vee (x_2 \wedge x_3 \wedge x_4).$$

The Boolean System Calculator, a function in the Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm), may yield the reliability index. Assume the 5 independent uncertain elements have reliabilities

$$0.91,\ 0.92,\ 0.93,\ 0.94,\ 0.95$$

in uncertain measure. A run of Boolean System Calculator shows that the reliability index is

$$\textit{Reliability} = \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_5) = 1\} = 0.92$$

in uncertain measure.

6.7 Bibliographic Notes

Uncertain reliability analysis was proposed by Liu [128] in 2010 in which a reliability index was defined and a reliability index theorem was proved.

Chapter 7

Uncertain Propositional Logic

Propositional logic, originated from the work of Aristotle (384-322 BC), is a branch of logic that studies the properties of complex propositions composed of simpler propositions and logical connectives. Note that the propositions considered in propositional logic are not arbitrary statements but are the ones that are either true or false and not both.

Uncertain propositional logic is a generalization of propositional logic in which every proposition is abstracted into a Boolean uncertain variable and the truth value is defined as the uncertain measure that the proposition is true. This chapter will deal with uncertain propositional logic, including uncertain proposition, truth value definition, and truth value theorem. This chapter will also introduce uncertain predicate logic.

7.1 Uncertain Proposition

Definition 7.1 *(Li and Liu [100]) An uncertain proposition is a statement whose truth value is quantified by an uncertain measure.*

That is, if we use X to express an uncertain proposition and use α to express its truth value in uncertain measure, then the uncertain proposition X is essentially a Boolean uncertain variable

$$X = \begin{cases} 1 \text{ with uncertain measure } \alpha \\ 0 \text{ with uncertain measure } 1-\alpha \end{cases} \tag{7.1}$$

where $X = 1$ means X is true and $X = 0$ means X is false.

Example 7.1: "Tom is tall with truth value 0.7" is an uncertain proposition, where "Tom is tall" is a statement, and its truth value is 0.7 in uncertain measure.

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5_8

Example 7.2: "John is young with truth value 0.8" is an uncertain proposition, where "John is young" is a statement, and its truth value is 0.8 in uncertain measure.

Example 7.3: "Beijing is a big city with truth value 0.9" is an uncertain proposition, where "Beijing is a big city" is a statement, and its truth value is 0.9 in uncertain measure.

Connective Symbols

In addition to the proposition symbols X and Y, we also need the negation symbol $\neg$, conjunction symbol $\wedge$, disjunction symbol $\vee$, conditional symbol $\rightarrow$, and biconditional symbol $\leftrightarrow$. Note that

$$\neg X \text{ means "not } X\text{";} \tag{7.2}$$

$$X \wedge Y \text{ means "}X \text{ and } Y\text{";} \tag{7.3}$$

$$X \vee Y \text{ means "}X \text{ or } Y\text{";} \tag{7.4}$$

$$X \rightarrow Y = (\neg X) \vee Y \text{ means "if } X \text{ then } Y\text{",} \tag{7.5}$$

$$X \leftrightarrow Y = (X \rightarrow Y) \wedge (Y \rightarrow X) \text{ means "}X \text{ if and only if } Y\text{".} \tag{7.6}$$

Boolean Function of Uncertain Propositions

Assume $X_1, X_2, \cdots, X_n$ are uncertain propositions. Then their Boolean function

$$Z = f(X_1, X_2, \cdots, X_n) \tag{7.7}$$

is a Boolean uncertain variable. Thus Z is also an uncertain proposition provided that it makes sense. Usually, such a Boolean function is a finite sequence of uncertain propositions and connective symbols. For example,

$$Z = \neg X_1, \quad Z = X_1 \wedge (\neg X_2), \quad Z = X_1 \rightarrow X_2 \tag{7.8}$$

are all uncertain propositions.

Independence of Uncertain Propositions

Uncertain propositions are called *independent* if they are independent uncertain variables. Assume $X_1, X_2, \cdots, X_n$ are independent uncertain propositions. Then

$$f_1(X_1),\ f_2(X_2) \cdots, f_n(X_n) \tag{7.9}$$

are also independent uncertain propositions for any Boolean functions $f_1, f_2, \cdots, f_n$. For example, if $X_1, X_2, \cdots, X_5$ are independent uncertain propositions, then $\neg X_1, X_2 \vee X_3, X_4 \rightarrow X_5$ are also independent.

7.2 Truth Value

Truth value is a key concept in uncertain propositional logic, and is defined as the uncertain measure that the uncertain proposition is true.

Definition 7.2 *(Li and Liu [100]) Let X be an uncertain proposition. Then the truth value of X is defined as the uncertain measure that X is true, i.e.,*

$$T(X) = \mathcal{M}\{X = 1\}. \tag{7.10}$$

Example 7.4: Let X be an uncertain proposition with truth value α. Then

$$T(\neg X) = \mathcal{M}\{X = 0\} = 1 - \alpha. \tag{7.11}$$

Example 7.5: Let X and Y be two independent uncertain propositions with truth values α and β, respectively. Then

$$T(X \wedge Y) = \mathcal{M}\{X \wedge Y = 1\} = \mathcal{M}\{(X = 1) \cap (Y = 1)\} = \alpha \wedge \beta, \tag{7.12}$$

$$T(X \vee Y) = \mathcal{M}\{X \vee Y = 1\} = \mathcal{M}\{(X = 1) \cup (Y = 1)\} = \alpha \vee \beta, \tag{7.13}$$

$$T(X \to Y) = T(\neg X \vee Y) = (1 - \alpha) \vee \beta. \tag{7.14}$$

Theorem 7.1 *(Law of Excluded Middle) Let X be an uncertain proposition. Then $X \vee \neg X$ is a tautology, i.e.,*

$$T(X \vee \neg X) = 1. \tag{7.15}$$

Proof: It follows from the definition of truth value and the property of uncertain measure that

$$T(X \vee \neg X) = \mathcal{M}\{X \vee \neg X = 1\} = \mathcal{M}\{(X = 1) \cup (X = 0)\} = \mathcal{M}\{\Gamma\} = 1.$$

The theorem is proved.

Theorem 7.2 *(Law of Contradiction) Let X be an uncertain proposition. Then $X \wedge \neg X$ is a contradiction, i.e.,*

$$T(X \wedge \neg X) = 0. \tag{7.16}$$

Proof: It follows from the definition of truth value and the property of uncertain measure that

$$T(X \wedge \neg X) = \mathcal{M}\{X \wedge \neg X = 1\} = \mathcal{M}\{(X = 1) \cap (X = 0)\} = \mathcal{M}\{\emptyset\} = 0.$$

The theorem is proved.

Theorem 7.3 *(Law of Truth Conservation) Let X be an uncertain proposition. Then we have*

$$T(X) + T(\neg X) = 1. \tag{7.17}$$

Proof: It follows from the duality axiom of uncertain measure that

$$T(\neg X) = \mathcal{M}\{\neg X = 1\} = \mathcal{M}\{X = 0\} = 1 - \mathcal{M}\{X = 1\} = 1 - T(X).$$

The theorem is proved.

Theorem 7.4 *Let X be an uncertain proposition. Then $X \to X$ is a tautology, i.e.,*

$$T(X \to X) = 1. \tag{7.18}$$

Proof: It follows from the definition of conditional symbol and the law of excluded middle that

$$T(X \to X) = T(\neg X \vee X) = 1.$$

The theorem is proved.

Theorem 7.5 *Let X be an uncertain proposition. Then we have*

$$T(X \to \neg X) = 1 - T(X). \tag{7.19}$$

Proof: It follows from the definition of conditional symbol and the law of truth conservation that

$$T(X \to \neg X) = T(\neg X \vee \neg X) = T(\neg X) = 1 - T(X).$$

The theorem is proved.

Theorem 7.6 *(De Morgan's Law) For any uncertain propositions X and Y, we have*

$$T(\neg(X \wedge Y)) = T((\neg X) \vee (\neg Y)), \tag{7.20}$$

$$T(\neg(X \vee Y)) = T((\neg X) \wedge (\neg Y)). \tag{7.21}$$

Proof: It follows from the basic properties of uncertain measure that

$$\begin{aligned} T(\neg(X \wedge Y)) &= \mathcal{M}\{X \wedge Y = 0\} = \mathcal{M}\{(X = 0) \cup (Y = 0)\} \\ &= \mathcal{M}\{(\neg X) \vee (\neg Y) = 1\} = T((\neg X) \vee (\neg Y)) \end{aligned}$$

which proves the first equality. A similar way may verify the second equality.

Theorem 7.7 *(Law of Contraposition) For any uncertain propositions X and Y, we have*

$$T(X \to Y) = T(\neg Y \to \neg X). \tag{7.22}$$

Proof: It follows from the definition of conditional symbol and basic properties of uncertain measure that

$$\begin{aligned} T(X \to Y) &= \mathcal{M}\{(\neg X) \vee Y = 1\} = \mathcal{M}\{(X = 0) \cup (Y = 1)\} \\ &= \mathcal{M}\{Y \vee (\neg X) = 1\} = T(\neg Y \to \neg X). \end{aligned}$$

The theorem is proved.

7.3 Chen-Ralescu Theorem

An important contribution to uncertain propositional logic is the Chen-Ralescu theorem that provides a numerical method for calculating the truth values of uncertain propositions.

Theorem 7.8 *(Chen-Ralescu Theorem [14]) Assume that* $X_1, X_2, \cdots, X_n$ *are independent uncertain propositions with truth values* $\alpha_1, \alpha_2, \cdots, \alpha_n$*, respectively. Then for a Boolean function* f*, the uncertain proposition*

$$Z = f(X_1, X_2, \cdots, X_n). \tag{7.23}$$

has a truth value

$$T(Z) = \begin{cases} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \text{if} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \sup\limits_{f(x_1,x_2,\cdots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \text{if} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) \ge 0.5 \end{cases} \tag{7.24}$$

where x_i *take values either 0 or 1, and* ν_i *are defined by*

$$\nu_i(x_i) = \begin{cases} \alpha_i, & \text{if } x_i = 1 \\ 1 - \alpha_i, & \text{if } x_i = 0 \end{cases} \tag{7.25}$$

for $i = 1, 2, \cdots, n$*, respectively.*

Proof: Since $Z = 1$ if and only if $f(X_1, X_2, \cdots, X_n) = 1$, we immediately have

$$T(Z) = \mathcal{M}\{f(X_1, X_2, \cdots, X_n) = 1\}.$$

Thus the equation (7.24) follows from Theorem 2.23 immediately.

Exercise 7.1: Let $X_1, X_2, \cdots, X_n$ be independent uncertain propositions with truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$, respectively. Then

$$Z = X_1 \wedge X_2 \wedge \cdots \wedge X_n \tag{7.26}$$

is an uncertain proposition. Show that the truth value of Z is

$$T(Z) = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_n. \tag{7.27}$$

Exercise 7.2: Let $X_1, X_2, \cdots, X_n$ be independent uncertain propositions with truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$, respectively. Then

$$Z = X_1 \vee X_2 \vee \cdots \vee X_n \tag{7.28}$$

is an uncertain proposition. Show that the truth value of Z is

$$T(Z) = \alpha_1 \vee \alpha_2 \vee \cdots \vee \alpha_n. \tag{7.29}$$

Example 7.6: Let X_1 and X_2 be independent uncertain propositions with truth values α_1 and α_2, respectively. Then

$$Z = X_1 \leftrightarrow X_2 \tag{7.30}$$

is an uncertain proposition. It is clear that $Z = f(X_1, X_2)$ if we define

$$f(1,1) = 1, \quad f(1,0) = 0, \quad f(0,1) = 0, \quad f(0,0) = 1.$$

At first, we have

$$\sup_{f(x_1,x_2)=1} \min_{1\le i\le 2} \nu_i(x_i) = \max\{\alpha_1 \wedge \alpha_2, (1-\alpha_1) \wedge (1-\alpha_2)\},$$

$$\sup_{f(x_1,x_2)=0} \min_{1\le i\le 2} \nu_i(x_i) = \max\{(1-\alpha_1) \wedge \alpha_2, \alpha_1 \wedge (1-\alpha_2)\}.$$

When $\alpha_1 \ge 0.5$ and $\alpha_2 \ge 0.5$, we have

$$\sup_{f(x_1,x_2)=1} \min_{1\le i\le 2} \nu_i(x_i) = \alpha_1 \wedge \alpha_2 \ge 0.5.$$

It follows from Chen-Ralescu theorem that

$$T(Z) = 1 - \sup_{f(x_1,x_2)=0} \min_{1\le i\le 2} \nu_i(x_i) = 1 - (1-\alpha_1) \vee (1-\alpha_2) = \alpha_1 \wedge \alpha_2.$$

When $\alpha_1 \ge 0.5$ and $\alpha_2 < 0.5$, we have

$$\sup_{f(x_1,x_2)=1} \min_{1\le i\le 2} \nu_i(x_i) = (1-\alpha_1) \vee \alpha_2 \le 0.5.$$

It follows from Chen-Ralescu theorem that

$$T(Z) = \sup_{f(x_1,x_2)=1} \min_{1\le i\le 2} \nu_i(x_i) = (1-\alpha_1) \vee \alpha_2.$$

When $\alpha_1 < 0.5$ and $\alpha_2 \ge 0.5$, we have

$$\sup_{f(x_1,x_2)=1} \min_{1\le i\le 2} \nu_i(x_i) = \alpha_1 \vee (1-\alpha_2) \le 0.5.$$

It follows from Chen-Ralescu theorem that

$$T(Z) = \sup_{f(x_1,x_2)=1} \min_{1\le i\le 2} \nu_i(x_i) = \alpha_1 \vee (1-\alpha_2).$$

When $\alpha_1 < 0.5$ and $\alpha_2 < 0.5$, we have

$$\sup_{f(x_1,x_2)=1} \min_{1\le i\le 2} \nu_i(x_i) = (1-\alpha_1)\wedge(1-\alpha_2) > 0.5.$$

It follows from Chen-Ralescu theorem that

$$T(Z) = 1 - \sup_{f(x_1,x_2)=0} \min_{1\le i\le 2} \nu_i(x_i) = 1-\alpha_1\vee\alpha_2 = (1-\alpha_1)\wedge(1-\alpha_2).$$

Thus we have

$$T(Z) = \begin{cases} \alpha_1\wedge\alpha_2, & \text{if } \alpha_1\ge 0.5 \text{ and } \alpha_2\ge 0.5 \\ (1-\alpha_1)\vee\alpha_2, & \text{if } \alpha_1\ge 0.5 \text{ and } \alpha_2< 0.5 \\ \alpha_1\vee(1-\alpha_2), & \text{if } \alpha_1< 0.5 \text{ and } \alpha_2\ge 0.5 \\ (1-\alpha_1)\wedge(1-\alpha_2), & \text{if } \alpha_1< 0.5 \text{ and } \alpha_2< 0.5. \end{cases} \tag{7.31}$$

7.4 Boolean System Calculator

Boolean System Calculator is a software that may compute the truth value of uncertain formula. This software may be downloaded from the website at http://orsc.edu.cn/liu/resources.htm. For example, assume $\xi_1,\xi_2,\xi_3,\xi_4,\xi_5$ are independent uncertain propositions with truth values $0.1, 0.3, 0.5, 0.7, 0.9$, respectively. Consider an uncertain formula,

$$X = (\xi_1\wedge\xi_2)\vee(\xi_2\wedge\xi_3)\vee(\xi_3\wedge\xi_4)\vee(\xi_4\wedge\xi_5). \tag{7.32}$$

It is clear that the corresponding Boolean function of X has the form

$$f(x_1,x_2,x_3,x_4,x_5) = \begin{cases} 1, & \text{if } x_1+x_2=2 \\ 1, & \text{if } x_2+x_3=2 \\ 1, & \text{if } x_3+x_4=2 \\ 1, & \text{if } x_4+x_5=2 \\ 0, & \text{otherwise.} \end{cases}$$

A run of Boolean System Calculator shows that the truth value of X is 0.7 in uncertain measure.

7.5 Uncertain Predicate Logic

Consider the following propositions: "Beijing is a big city", and "Tianjin is a big city". Uncertain propositional logic treats them as unrelated propositions. However, uncertain predicate logic represents them by a predicate proposition $X(a)$. If a represents Beijing, then

$$X(a) = \text{“Beijing is a big city”}. \tag{7.33}$$

If a represents Tianjin, then

$$X(a) = \text{“Tianjin is a big city”}. \tag{7.34}$$

Definition 7.3 *(Zhang and Li [267]) Uncertain predicate proposition is a sequence of uncertain propositions indexed by one or more parameters.*

In order to deal with uncertain predicate propositions, we need a universal quantifier $\forall$ and an existential quantifier $\exists$. If $X(a)$ is an uncertain predicate proposition defined by (7.33) and (7.34), then

$$(\forall a)X(a) = \text{“Both Beijing and Tianjin are big cities”}, \tag{7.35}$$

$$(\exists a)X(a) = \text{“At least one of Beijing and Tianjin is a big city”}. \tag{7.36}$$

Theorem 7.9 *(Zhang and Li [267], Law of Excluded Middle) Let $X(a)$ be an uncertain predicate proposition. Then*

$$T((\forall a)X(a) \vee (\exists a)\neg X(a)) = 1. \tag{7.37}$$

Proof: Since $\neg(\forall a)X(a) = (\exists a)\neg X(a)$, it follows from the definition of truth value and the property of uncertain measure that

$$T((\forall a)X(a) \vee (\exists a)\neg X(a)) = \mathcal{M}\{((\forall a)X(a) = 1) \cup ((\forall a)X(a) = 0)\} = 1.$$

The theorem is proved.

Theorem 7.10 *(Zhang and Li [267], Law of Contradiction) Let $X(a)$ be an uncertain predicate proposition. Then*

$$T((\forall a)X(a) \wedge (\exists a)\neg X(a)) = 0. \tag{7.38}$$

Proof: Since $\neg(\forall a)X(a) = (\exists a)\neg X(a)$, it follows from the definition of truth value and the property of uncertain measure that

$$T((\forall a)X(a) \wedge (\exists a)\neg X(a)) = \mathcal{M}\{((\forall a)X(a) = 1) \cap ((\forall a)X(a) = 0)\} = 0.$$

The theorem is proved.

Theorem 7.11 *(Zhang and Li [267], Law of Truth Conservation) Let $X(a)$ be an uncertain predicate proposition. Then*

$$T((\forall a)X(a)) + T((\exists a)\neg X(a)) = 1. \tag{7.39}$$

Proof: Since $\neg(\forall a)X(a) = (\exists a)\neg X(a)$, it follows from the definition of truth value and the property of uncertain measure that

$$T((\exists a)\neg X(a)) = 1 - \mathcal{M}\{(\forall a)X(a) = 1\} = 1 - T((\forall a)X(a)).$$

The theorem is proved.

Theorem 7.12 *(Zhang and Li [267]) Let $X(a)$ be an uncertain predicate proposition. Then for any given b, we have*

$$T((\forall a)X(a) \to X(b)) = 1. \tag{7.40}$$

Proof: The argument breaks into two cases. Case 1: If $X(b) = 0$, then $(\forall a)X(a) = 0$ and $\neg(\forall a)X(a) = 1$. Thus

$$(\forall a)X(a) \to X(b) = \neg(\forall a)X(a) \vee X(b) = 1.$$

Case II: If $X(b) = 1$, then we immediately have

$$(\forall a)X(a) \to X(b) = \neg(\forall a)X(a) \vee X(b) = 1.$$

Thus we always have (7.40). The theorem is proved.

Theorem 7.13 *(Zhang and Li [267]) Let $X(a)$ be an uncertain predicate proposition. Then for any given b, we have*

$$T(X(b) \to (\exists a)X(a)) = 1. \tag{7.41}$$

Proof: The argument breaks into two cases. Case 1: If $X(b) = 0$, then $\neg X(b) = 1$ and

$$X(b) \to (\forall a)X(a) = \neg X(b) \vee (\exists a)X(a) = 1.$$

Case II: If $X(b) = 1$, then $(\exists a)X(a) = 1$ and

$$X(b) \to (\exists a)X(a) = \neg X(b) \vee (\exists a)X(a) = 1.$$

Thus we always have (7.41). The theorem is proved.

Theorem 7.14 *(Zhang and Li [267]) Let $X(a)$ be an uncertain predicate proposition. Then*

$$T((\forall a)X(a) \to (\exists a)X(a)) = 1. \tag{7.42}$$

Proof: The argument breaks into two cases. Case 1: If $(\forall a)X(a) = 0$, then $\neg(\forall a)X(a) = 1$ and

$$(\forall a)X(a) \to (\exists a)X(a) = \neg(\forall a)X(a) \vee (\exists a)X(a) = 1.$$

Case II: If $(\forall a)X(a) = 1$, then $(\exists a)X(a) = 1$ and

$$(\forall a)X(a) \to (\exists a)X(a) = \neg(\forall a)X(a) \vee (\exists a)X(a) = 1.$$

Thus we always have (7.42). The theorem is proved.

Theorem 7.15 *(Zhang and Li [267]) Let $X(a)$ be an uncertain predicate proposition such that $\{X(a)|a \in A\}$ is a class of independent uncertain propositions. Then*

$$T((\forall a)X(a)) = \inf_{a\in A} T(X(a)), \tag{7.43}$$

$$T((\exists a)X(a)) = \sup_{a\in A} T(X(a)). \tag{7.44}$$

Proof: For each uncertain predicate proposition $X(a)$, by the meaning of universal quantifier, we obtain

$$T((\forall a)X(a)) = \mathcal{M}\{(\forall a)X(a) = 1\} = \mathcal{M}\left\{\bigcap_{a\in A}(X(a) = 1)\right\}.$$

Since $\{X(a)|a \in A\}$ is a class of independent uncertain propositions, we get

$$T((\forall a)X(a)) = \inf_{a\in A} \mathcal{M}\{X(a) = 1\} = \inf_{a\in A} T(X(a)).$$

The first equation is verified. Similarly, by the meaning of existential quantifier, we obtain

$$T((\exists a)X(a)) = \mathcal{M}\{(\exists a)X(a) = 1\} = \mathcal{M}\left\{\bigcup_{a\in A}(X(a) = 1)\right\}.$$

Since $\{X(a)|a \in A\}$ is a class of independent uncertain propositions, we get

$$T((\exists a)X(a)) = \sup_{a\in A} \mathcal{M}\{X(a) = 1\} = \sup_{a\in A} T(X(a)).$$

The second equation is proved.

Theorem 7.16 *(Zhang and Li [267]) Let $X(a, b)$ be an uncertain predicate proposition such that $\{X(a, b)|a \in A, b \in B\}$ is a class of independent uncertain propositions. Then*

$$T((\forall a)(\exists b)X(a, b)) = \inf_{a\in A} \sup_{b\in B} T(X(a, b)), \tag{7.45}$$

$$T((\exists a)(\forall b)X(a, b)) = \sup_{a\in A} \inf_{b\in B} T(X(a, b)). \tag{7.46}$$

Proof: Since $\{X(a, b)|a \in A, b \in B\}$ is a class of independent uncertain propositions, both $\{(\exists b)X(a, b)|a \in A\}$ and $\{(\forall b)X(a, b)|a \in A\}$ are two classes of independent uncertain propositions. It follows from Theorem 7.15 that

$$T((\forall a)(\exists b)X(a, b)) = \inf_{a\in A} T((\exists b)X(a, b)) = \inf_{a\in A} \sup_{b\in B} T(X(a, b)),$$

$$T((\exists a)(\forall b)X(a, b)) = \sup_{a\in A} T((\forall b)X(a, b)) = \sup_{a\in A} \inf_{b\in B} T(X(a, b)).$$

The theorem is proved.

7.6 Bibliographic Notes

Uncertain propositional logic was designed by Li and Liu [100] in which every proposition is abstracted into a Boolean uncertain variable and the truth value is defined as the uncertain measure that the proposition is true. An important contribution is Chen-Ralescu theorem [14] that provides a numerical method for calculating the truth value of uncertain propositions.

Another topic is the uncertain predicate logic developed by Zhang and Li [267] in which an uncertain predicate proposition is defined as a sequence of uncertain propositions indexed by one or more parameters.

Chapter 8

Uncertain Entailment

Uncertain entailment is a methodology for calculating the truth value of an uncertain formula via the maximum uncertainty principle when the truth values of other uncertain formulas are given. In some sense, uncertain propositional logic and uncertain entailment are mutually inverse, the former attempts to compose a complex proposition from simpler ones, while the latter attempts to decompose a complex proposition into simpler ones.

This chapter will present an uncertain entailment model. In addition, uncertain modus ponens, uncertain modus tollens and uncertain hypothetical syllogism are deduced from the uncertain entailment model.

8.1 Uncertain Entailment Model

Assume $X_1, X_2, \cdots, X_n$ are independent uncertain propositions with *unknown* truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$, respectively. Also assume that

$$Y_j = f_j(X_1, X_2, \cdots, X_n) \tag{8.1}$$

are uncertain propositions with *known* truth values c_j, $j = 1, 2, \cdots, m$, respectively. Now let

$$Z = f(X_1, X_2, \cdots, X_n) \tag{8.2}$$

be an additional uncertain proposition. What is the truth value of Z? This is just the uncertain entailment problem. In order to solve it, let us consider what values $\alpha_1, \alpha_2, \cdots, \alpha_n$ may take. The first constraint is

$$0 \le \alpha_i \le 1, \quad i = 1, 2, \cdots, n. \tag{8.3}$$

The second type of constraints is represented by

$$T(Y_j) = c_j \tag{8.4}$$

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5_9

where $T(Y_j)$ are determined by $\alpha_1, \alpha_2, \cdots, \alpha_n$ via

$$T(Y_j) = \begin{cases} \sup\limits_{f_j(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \text{if } \sup\limits_{f_j(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \sup\limits_{f_j(x_1,x_2,\cdots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \text{if } \sup\limits_{f_j(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) \ge 0.5 \end{cases} \tag{8.5}$$

for $j = 1, 2, \cdots, m$ and

$$\nu_i(x_i) = \begin{cases} \alpha_i, & \text{if } x_i = 1 \\ 1 - \alpha_i, & \text{if } x_i = 0 \end{cases} \tag{8.6}$$

for $i = 1, 2, \cdots, n$. Please note that the additional uncertain proposition $Z = f(X_1, X_2, \cdots, X_n)$ has a truth value

$$T(Z) = \begin{cases} \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \text{if } \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \sup\limits_{f(x_1,x_2,\cdots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \text{if } \sup\limits_{f(x_1,x_2,\cdots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) \ge 0.5. \end{cases} \tag{8.7}$$

Since the truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$ are not uniquely determined, the truth value $T(Z)$ is not unique too. In this case, we have to use the maximum uncertainty principle to determine the truth value $T(Z)$. That is, $T(Z)$ should be assigned the value as close to 0.5 as possible. In other words, we should minimize the value $|T(Z) - 0.5|$ via choosing appreciate values of $\alpha_1, \alpha_2, \cdots, \alpha_n$. The uncertain entailment model is thus written by Liu [126] as follows,

$$\begin{cases} \min |T(Z) - 0.5| \\ \text{subject to:} \\ \qquad 0 \le \alpha_i \le 1, \quad i = 1, 2, \cdots, n \\ \qquad T(Y_j) = c_j, \quad j = 1, 2, \cdots, m \end{cases} \tag{8.8}$$

where $T(Z), T(Y_j), j = 1, 2, \cdots, m$ are functions of unknown truth values $\alpha_1, \alpha_2, \cdots, \alpha_n$.

Example 8.1: Let A and B be independent uncertain propositions. It is known that

$$T(A \vee B) = a, \quad T(A \wedge B) = b. \tag{8.9}$$

What is the truth value of $A \to B$? Denote the truth values of A and B by α_1 and α_2, respectively, and write

$$Y_1 = A \vee B, \quad Y_2 = A \wedge B, \quad Z = A \to B.$$

It is clear that

$$T(Y_1) = \alpha_1 \vee \alpha_2 = a,$$
$$T(Y_2) = \alpha_1 \wedge \alpha_2 = b,$$
$$T(Z) = (1 - \alpha_1) \vee \alpha_2.$$

In this case, the uncertain entailment model (8.8) becomes

$$\begin{cases} \min |(1-\alpha_1) \vee \alpha_2 - 0.5| \\ \text{subject to:} \\ \qquad 0 \le \alpha_1 \le 1 \\ \qquad 0 \le \alpha_2 \le 1 \\ \qquad \alpha_1 \vee \alpha_2 = a \\ \qquad \alpha_1 \wedge \alpha_2 = b. \end{cases} \tag{8.10}$$

When $a \ge b$, there are only two feasible solutions $(\alpha_1, \alpha_2) = (a, b)$ and $(\alpha_1, \alpha_2) = (b, a)$. If $a + b < 1$, the optimal solution produces

$$T(Z) = (1 - \alpha_1^*) \vee \alpha_2^* = 1 - a;$$

if $a + b = 1$, the optimal solution produces

$$T(Z) = (1 - \alpha_1^*) \vee \alpha_2^* = a \text{ or } b;$$

if $a + b > 1$, the optimal solution produces

$$T(Z) = (1 - \alpha_1^*) \vee \alpha_2^* = b.$$

When $a < b$, there is no feasible solution and the truth values are ill-assigned. In summary, from $T(A \vee B) = a$ and $T(A \wedge B) = b$ we entail

$$T(A \to B) = \begin{cases} 1 - a, & \text{if } a \ge b \text{ and } a + b < 1 \\ a \text{ or } b, & \text{if } a \ge b \text{ and } a + b = 1 \\ b, & \text{if } a \ge b \text{ and } a + b > 1 \\ \text{illness}, & \text{if } a < b. \end{cases} \tag{8.11}$$

8.2 Uncertain Modus Ponens

Uncertain modus ponens was presented by Liu [126]. Let A and B be independent uncertain propositions. Assume A and $A \to B$ have truth values a

and b, respectively. What is the truth value of B? Denote the truth values of A and B by α_1 and α_2, respectively, and write

$$Y_1 = A, \quad Y_2 = A \to B, \quad Z = B.$$

It is clear that

$$T(Y_1) = \alpha_1 = a,$$

$$T(Y_2) = (1 - \alpha_1) \vee \alpha_2 = b,$$

$$T(Z) = \alpha_2.$$

In this case, the uncertain entailment model (8.8) becomes

$$\begin{cases} \min |\alpha_2 - 0.5| \\ \text{subject to:} \\ \qquad 0 \le \alpha_1 \le 1 \\ \qquad 0 \le \alpha_2 \le 1 \\ \qquad \alpha_1 = a \\ \qquad (1 - \alpha_1) \vee \alpha_2 = b. \end{cases} \tag{8.12}$$

When $a + b > 1$, there is a unique feasible solution and then the optimal solution is

$$\alpha_1^* = a, \quad \alpha_2^* = b.$$

Thus $T(B) = \alpha_2^* = b$. When $a + b = 1$, the feasible set is $\{a\} \times [0, b]$ and the optimal solution is

$$\alpha_1^* = a, \quad \alpha_2^* = 0.5 \wedge b.$$

Thus $T(B) = \alpha_2^* = 0.5 \wedge b$. When $a + b < 1$, there is no feasible solution and the truth values are ill-assigned. In summary, from

$$T(A) = a, \quad T(A \to B) = b \tag{8.13}$$

we entail

$$T(B) = \begin{cases} b, & \text{if } a + b > 1 \\ 0.5 \wedge b, & \text{if } a + b = 1 \\ \text{illness}, & \text{if } a + b < 1. \end{cases} \tag{8.14}$$

This result coincides with the classical modus ponens that if both A and $A \to B$ are true, then B is true.

8.3 Uncertain Modus Tollens

Uncertain modus tollens was presented by Liu [126]. Let A and B be independent uncertain propositions. Assume $A \to B$ and B have truth values a

and b, respectively. What is the truth value of A? Denote the truth values of A and B by α_1 and α_2, respectively, and write

$$Y_1 = A \to B, \quad Y_2 = B, \quad Z = A.$$

It is clear that

$$T(Y_1) = (1 - \alpha_1) \vee \alpha_2 = a,$$

$$T(Y_2) = \alpha_2 = b,$$

$$T(Z) = \alpha_1.$$

In this case, the uncertain entailment model (8.8) becomes

$$\begin{cases} \min |\alpha_1 - 0.5| \\ \text{subject to:} \\ \qquad 0 \le \alpha_1 \le 1 \\ \qquad 0 \le \alpha_2 \le 1 \\ \qquad (1 - \alpha_1) \vee \alpha_2 = a \\ \qquad \alpha_2 = b. \end{cases} \tag{8.15}$$

When $a > b$, there is a unique feasible solution and then the optimal solution is

$$\alpha_1^* = 1 - a, \quad \alpha_2^* = b.$$

Thus $T(A) = \alpha_1^* = 1 - a$. When $a = b$, the feasible set is $[1 - a, 1] \times \{b\}$ and the optimal solution is

$$\alpha_1^* = (1 - a) \vee 0.5, \quad \alpha_2^* = b.$$

Thus $T(A) = \alpha_1^* = (1 - a) \vee 0.5$. When $a < b$, there is no feasible solution and the truth values are ill-assigned. In summary, from

$$T(A \to B) = a, \quad T(B) = b \tag{8.16}$$

we entail

$$T(A) = \begin{cases} 1 - a, & \text{if } a > b \\ (1 - a) \vee 0.5, & \text{if } a = b \\ \text{illness}, & \text{if } a < b. \end{cases} \tag{8.17}$$

This result coincides with the classical modus tollens that if $A \to B$ is true and B is false, then A is false.

8.4 Uncertain Hypothetical Syllogism

Uncertain hypothetical syllogism was presented by Liu [126]. Let A, B, C be independent uncertain propositions. Assume $A \to B$ and $B \to C$ have truth values a and b, respectively. What is the truth value of $A \to C$? Denote the truth values of A, B, C by $\alpha_1, \alpha_2, \alpha_3$, respectively, and write

$$Y_1 = A \to B, \quad Y_2 = B \to C, \quad Z = A \to C.$$

It is clear that

$$\begin{aligned} T(Y_1) &= (1-\alpha_1) \vee \alpha_2 = a, \\ T(Y_2) &= (1-\alpha_2) \vee \alpha_3 = b, \\ T(Z) &= (1-\alpha_1) \vee \alpha_3. \end{aligned}$$

In this case, the uncertain entailment model (8.8) becomes

$$\left\{\begin{array}{l} \min |(1-\alpha_1) \vee \alpha_3 - 0.5| \\ \text{subject to:} \\ \qquad 0 \le \alpha_1 \le 1 \\ \qquad 0 \le \alpha_2 \le 1 \\ \qquad 0 \le \alpha_3 \le 1 \\ \qquad (1-\alpha_1) \vee \alpha_2 = a \\ \qquad (1-\alpha_2) \vee \alpha_3 = b. \end{array}\right. \tag{8.18}$$

Write the optimal solution by $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$. When $a \wedge b \ge 0.5$, we have

$$T(A \to C) = (1-\alpha_1^*) \vee \alpha_3^* = a \wedge b.$$

When $a + b \ge 1$ and $a \wedge b < 0.5$, we have

$$T(A \to C) = (1-\alpha_1^*) \vee \alpha_3^* = 0.5.$$

When $a + b < 1$, there is no feasible solution and the truth values are ill-assigned. In summary, from

$$T(A \to B) = a, \quad T(B \to C) = b \tag{8.19}$$

we entail

$$T(A \to C) = \begin{cases} a \wedge b, & \text{if } a \ge 0.5 \text{ and } b \ge 0.5 \\ 0.5, & \text{if } a + b \ge 1 \text{ and } a \wedge b < 0.5 \\ \text{illness}, & \text{if } a + b < 1. \end{cases} \tag{8.20}$$

This result coincides with the classical hypothetical syllogism that if both $A \to B$ and $B \to C$ are true, then $A \to C$ is true.

8.5 Bibliographic Notes

Uncertain entailment was proposed by Liu [126] for determining the truth value of an uncertain proposition via the maximum uncertainty principle when the truth values of other uncertain propositions are given. From the uncertain entailment model, Liu [126] also deduced uncertain modus ponens, uncertain modus tollens, and uncertain hypothetical syllogism.

Chapter 9

Uncertain Set

Uncertain set was first proposed by Liu [127] in 2010 for modeling unsharp concepts. This chapter will introduce the concepts of uncertain set, membership function, independence, expected value, variance, entropy, and distance. This chapter will also introduce the operational law for uncertain sets via membership functions or inverse membership functions, and uncertain statistics for determining membership functions.

9.1 Uncertain Set

Roughly speaking, an uncertain set is a set-valued function on an uncertainty space, and attempts to model "unsharp concepts" that are essentially sets but their boundaries are not sharply described (because of the ambiguity of human language). Some typical examples include "young", "tall", "warm", and "most". A formal definition is given as follows.

Definition 9.1 *(Liu [127]) An uncertain set is a function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to a collection of sets of real numbers such that both $\{B \subset \xi\}$ and $\{\xi \subset B\}$ are events for any Borel set B.*

Remark 9.1: It is clear that uncertain set (Liu [127]) is very different from random set (Robbins [198] and Matheron [167]) and fuzzy set (Zadeh [260]). The essential difference among them is that different measures are used, i.e., random set uses probability measure, fuzzy set uses possibility measure and uncertain set uses uncertain measure.

Remark 9.2: What is the difference between uncertain variable and uncertain set? Both of them belong to the same broad category of uncertain concepts. However, they are differentiated by their mathematical definitions: the former refers to one value, while the latter to a collection of values. Essentially, the difference between uncertain variable and uncertain set focuses

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5_10

on the property of *exclusivity*. If the concept has exclusivity, then it is an uncertain variable. Otherwise, it is an uncertain set. Consider the statement "John is a young man". If we are interested in John's real age, then "young" is an uncertain variable because it is an exclusive concept (John's age cannot be more than one value). For example, if John is 20 years old, then it is impossible that John is 25 years old. In other words, "John is 20 years old" does exclude the possibility that "John is 25 years old". By contrast, if we are interested in what ages can be regarded "young", then "young" is an uncertain set because the concept now has no exclusivity. For example, both 20-year-old and 25-year-old men can be considered "young". In other words, "a 20-year-old man is young" does not exclude the possibility that "a 25-year-old man is young".

Example 9.1: Take an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to be $\{\gamma_1,\gamma_2,\gamma_3\}$ with power set $\mathcal{L}$. Then the set-valued function

$$\xi(\gamma)=\begin{cases}[1,3], & \text{if } \gamma=\gamma_1\\ [2,4], & \text{if } \gamma=\gamma_2\\ [3,5], & \text{if } \gamma=\gamma_3\end{cases} \tag{9.1}$$

is an uncertain set on $(\Gamma,\mathcal{L},\mathcal{M})$. See Figure 9.1.

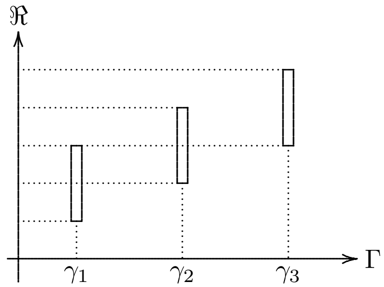


Figure 9.1: An Uncertain Set

Example 9.2: Take an uncertainty space $(\Gamma,\mathcal{L},\mathcal{M})$ to be $\Re$ with Borel algebra $\mathcal{L}$. Then the set-valued function

$$\xi(\gamma)=[\gamma,\gamma+1],\quad \forall\gamma\in\Gamma \tag{9.2}$$

is an uncertain set on $(\Gamma,\mathcal{L},\mathcal{M})$.

Theorem 9.1 *Let ξ be an uncertain set and let B be a Borel set. Then the set*

$$\{B\not\subset\xi\}=\{\gamma\in\Gamma \mid B\not\subset\xi(\gamma)\} \tag{9.3}$$

is an event.

Proof: Since ξ is an uncertain set and B is a Borel set, the set $\{B \subset \xi\}$ is an event. Thus $\{B \not\subset \xi\}$ is an event by using the relation $\{B \not\subset \xi\} = \{B \subset \xi\}^c$.

Theorem 9.2 *Let ξ be an uncertain set and let B be a Borel set. Then the set*

$$\{\xi \not\subset B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \not\subset B\} \tag{9.4}$$

is an event.

Proof: Since ξ is an uncertain set and B is a Borel set, the set $\{\xi \subset B\}$ is an event. Thus $\{\xi \not\subset B\}$ is an event by using the relation $\{\xi \not\subset B\} = \{\xi \subset B\}^c$.

Union, Intersection and Complement

Definition 9.2 *Let ξ and η be two uncertain sets on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$. Then (i) the union $\xi \cup \eta$ of the uncertain sets ξ and η is*

$$(\xi \cup \eta)(\gamma) = \xi(\gamma) \cup \eta(\gamma), \quad \forall \gamma \in \Gamma; \tag{9.5}$$

(ii) the intersection $\xi \cap \eta$ of the uncertain sets ξ and η is

$$(\xi \cap \eta)(\gamma) = \xi(\gamma) \cap \eta(\gamma), \quad \forall \gamma \in \Gamma; \tag{9.6}$$

(iii) the complement ξ^c of the uncertain set ξ is

$$\xi^c(\gamma) = \xi(\gamma)^c, \quad \forall \gamma \in \Gamma. \tag{9.7}$$

Example 9.3: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3\}$. Let ξ and η be two uncertain sets,

$$\xi(\gamma) = \begin{cases} [1,2], & \text{if } \gamma = \gamma_1 \\ [1,3], & \text{if } \gamma = \gamma_2 \\ [1,4], & \text{if } \gamma = \gamma_3, \end{cases} \qquad \eta(\gamma) = \begin{cases} (2,3), & \text{if } \gamma = \gamma_1 \\ (2,4), & \text{if } \gamma = \gamma_2 \\ (2,5), & \text{if } \gamma = \gamma_3. \end{cases}$$

Then their union is

$$(\xi \cup \eta)(\gamma) = \begin{cases} [1,3), & \text{if } \gamma = \gamma_1 \\ [1,4), & \text{if } \gamma = \gamma_2 \\ [1,5), & \text{if } \gamma = \gamma_3, \end{cases}$$

their intersection is

$$(\xi \cap \eta)(\gamma) = \begin{cases} \emptyset, & \text{if } \gamma = \gamma_1 \\ (2,3], & \text{if } \gamma = \gamma_2 \\ (2,4], & \text{if } \gamma = \gamma_3, \end{cases}$$

and their complements are

$$\xi^c(\gamma) = \begin{cases} (-\infty, 1) \cup (2, +\infty), & \text{if } \gamma = \gamma_1 \\ (-\infty, 1) \cup (3, +\infty), & \text{if } \gamma = \gamma_2 \\ (-\infty, 1) \cup (4, +\infty), & \text{if } \gamma = \gamma_3, \end{cases}$$

$$\eta^c(\gamma) = \begin{cases} (-\infty, 2] \cup [3, +\infty), & \text{if } \gamma = \gamma_1 \\ (-\infty, 2] \cup [4, +\infty), & \text{if } \gamma = \gamma_2 \\ (-\infty, 2] \cup [5, +\infty), & \text{if } \gamma = \gamma_3. \end{cases}$$

Theorem 9.3 *Let ξ be an uncertain set and let $\Re$ be the set of real numbers. Then*

$$\xi \cup \Re = \Re, \quad \xi \cap \Re = \xi. \tag{9.8}$$

Proof: For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that the union is

$$(\xi \cup \Re)(\gamma) = \xi(\gamma) \cup \Re = \Re.$$

Thus we have $\xi \cup \Re = \Re$. In addition, the intersection is

$$(\xi \cap \Re)(\gamma) = \xi(\gamma) \cap \Re = \xi(\gamma).$$

Thus we have $\xi \cap \Re = \xi$.

Theorem 9.4 *Let ξ be an uncertain set and let $\emptyset$ be the empty set. Then*

$$\xi \cup \emptyset = \xi, \quad \xi \cap \emptyset = \emptyset. \tag{9.9}$$

Proof: For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that the union is

$$(\xi \cup \emptyset)(\gamma) = \xi(\gamma) \cup \emptyset = \xi(\gamma).$$

Thus we have $\xi \cup \emptyset = \xi$. In addition, the intersection is

$$(\xi \cap \emptyset)(\gamma) = \xi(\gamma) \cap \emptyset = \emptyset.$$

Thus we have $\xi \cap \emptyset = \emptyset$.

Theorem 9.5 *(Idempotent Law) Let ξ be an uncertain set. Then we have*

$$\xi \cup \xi = \xi, \quad \xi \cap \xi = \xi. \tag{9.10}$$

Proof: For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that the union is

$$(\xi \cup \xi)(\gamma) = \xi(\gamma) \cup \xi(\gamma) = \xi(\gamma).$$

Thus we have $\xi \cup \xi = \xi$. In addition, the intersection is

$$(\xi \cap \xi)(\gamma) = \xi(\gamma) \cap \xi(\gamma) = \xi(\gamma).$$

Thus we have $\xi \cap \xi = \xi$.

Theorem 9.6 *(Double-Negation Law) Let ξ be an uncertain set. Then we have*

$$(\xi^c)^c = \xi. \tag{9.11}$$

Proof: For each $\gamma \in \Gamma$, it follows from the definition of complement that

$$(\xi^c)^c(\gamma) = (\xi^c(\gamma))^c = (\xi(\gamma)^c)^c = \xi(\gamma).$$

Thus we have $(\xi^c)^c = \xi$.

Theorem 9.7 *(Law of Excluded Middle and Law of Contradiction) Let ξ be an uncertain set and let ξ^c be its complement. Then*

$$\xi \cup \xi^c = \Re, \quad \xi \cap \xi^c = \emptyset. \tag{9.12}$$

Proof: For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that the union is

$$(\xi \cup \xi^c)(\gamma) = \xi(\gamma) \cup \xi^c(\gamma) = \xi(\gamma) \cup \xi(\gamma)^c = \Re.$$

Thus we have $\xi \cup \xi^c \equiv \Re$. In addition, the intersection is

$$(\xi \cap \xi^c)(\gamma) = \xi(\gamma) \cap \xi^c(\gamma) = \xi(\gamma) \cap \xi(\gamma)^c = \emptyset.$$

Thus we have $\xi \cap \xi^c \equiv \emptyset$.

Theorem 9.8 *(Commutative Law) Let ξ and η be uncertain sets. Then we have*

$$\xi \cup \eta = \eta \cup \xi, \quad \xi \cap \eta = \eta \cap \xi. \tag{9.13}$$

Proof: For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that

$$(\xi \cup \eta)(\gamma) = \xi(\gamma) \cup \eta(\gamma) = \eta(\gamma) \cup \xi(\gamma) = (\eta \cup \xi)(\gamma).$$

Thus we have $\xi \cup \eta = \eta \cup \xi$. In addition, it follows that

$$(\xi \cap \eta)(\gamma) = \xi(\gamma) \cap \eta(\gamma) = \eta(\gamma) \cap \xi(\gamma) = (\eta \cap \xi)(\gamma).$$

Thus we have $\xi \cap \eta = \eta \cap \xi$.

Theorem 9.9 *(Associative Law) Let ξ, η, τ be uncertain sets. Then we have*

$$(\xi \cup \eta) \cup \tau = \xi \cup (\eta \cup \tau), \quad (\xi \cap \eta) \cap \tau = \xi \cap (\eta \cap \tau). \tag{9.14}$$

Proof: For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that

$$\begin{aligned}((\xi \cup \eta) \cup \tau)(\gamma) &= (\xi(\gamma) \cup \eta(\gamma)) \cup \tau(\gamma) \\ &= \xi(\gamma) \cup (\eta(\gamma) \cup \tau(\gamma)) = (\xi \cup (\eta \cup \tau))(\gamma).\end{aligned}$$

Thus we have $(\xi \cup \eta) \cup \tau = \xi \cup (\eta \cup \tau)$. In addition, it follows that

$$\begin{aligned}((\xi \cap \eta) \cap \tau)(\gamma) &= (\xi(\gamma) \cap \eta(\gamma)) \cap \tau(\gamma) \\ &= \xi(\gamma) \cap (\eta(\gamma) \cap \tau(\gamma)) = (\xi \cap (\eta \cap \tau))(\gamma).\end{aligned}$$

Thus we have $(\xi \cap \eta) \cap \tau = \xi \cap (\eta \cap \tau)$.

Theorem 9.10 *(Distributive Law) Let ξ, η, τ be uncertain sets. Then we have*

$$\xi \cup (\eta \cap \tau) = (\xi \cup \eta) \cap (\xi \cup \tau), \quad \xi \cap (\eta \cup \tau) = (\xi \cap \eta) \cup (\xi \cap \tau). \tag{9.15}$$

Proof: For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that

$$\begin{aligned}(\xi \cup (\eta \cap \tau))(\gamma) &= \xi(\gamma) \cup (\eta(\gamma) \cap \tau(\gamma)) \\ &= (\xi(\gamma) \cup \eta(\gamma)) \cap (\xi(\gamma) \cup \tau(\gamma)) \\ &= ((\xi \cup \eta) \cap (\xi \cup \tau))(\gamma).\end{aligned}$$

Thus we have $\xi \cup (\eta \cap \tau) = (\xi \cup \eta) \cap (\xi \cup \tau)$. In addition, it follows that

$$\begin{aligned}(\xi \cap (\eta \cup \tau))(\gamma) &= \xi(\gamma) \cap (\eta(\gamma) \cup \tau(\gamma)) \\ &= (\xi(\gamma) \cap \eta(\gamma)) \cup (\xi(\gamma) \cap \tau(\gamma)) \\ &= ((\xi \cap \eta) \cup (\xi \cap \tau))(\gamma).\end{aligned}$$

Thus we have $\xi \cap (\eta \cup \tau) = (\xi \cap \eta) \cup (\xi \cap \tau)$.

Theorem 9.11 *(Absorbtion Law) Let ξ and η be uncertain sets. Then we have*

$$\xi \cup (\xi \cap \eta) = \xi, \quad \xi \cap (\xi \cup \eta) = \xi. \tag{9.16}$$

Proof: For each $\gamma \in \Gamma$, it follows from the definition of uncertain set that

$$(\xi \cup (\xi \cap \eta))(\gamma) = \xi(\gamma) \cup (\xi(\gamma) \cap \eta(\gamma)) = \xi(\gamma).$$

Thus we have $\xi \cup (\xi \cap \eta) = \xi$. In addition, since

$$(\xi \cap (\xi \cup \eta))(\gamma) = \xi(\gamma) \cap (\xi(\gamma) \cup \eta(\gamma)) = \xi(\gamma),$$

we get $\xi \cap (\xi \cup \eta) = \xi$.

Theorem 9.12 *(De Morgan's Law) Let ξ and η be uncertain sets. Then*

$$(\xi \cup \eta)^c = \xi^c \cap \eta^c, \quad (\xi \cap \eta)^c = \xi^c \cup \eta^c. \tag{9.17}$$

Proof: For each $\gamma \in \Gamma$, it follows from the definition of complement that

$$(\xi \cup \eta)^c(\gamma) = ((\xi(\gamma) \cup \eta(\gamma))^c = \xi(\gamma)^c \cap \eta(\gamma)^c = (\xi^c \cap \eta^c)(\gamma).$$

Thus we have $(\xi \cup \eta)^c = \xi^c \cap \eta^c$. In addition, since

$$(\xi \cap \eta)^c(\gamma) = ((\xi(\gamma) \cap \eta(\gamma))^c = \xi(\gamma)^c \cup \eta(\gamma)^c = (\xi^c \cup \eta^c)(\gamma),$$

we get $(\xi \cap \eta)^c = \xi^c \cup \eta^c$.

Function of Uncertain Sets

Definition 9.3 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain sets on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$, and let f be a measurable function. Then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain set defined by*

$$\xi(\gamma) = f(\xi_1(\gamma), \xi_2(\gamma), \cdots, \xi_n(\gamma)), \quad \forall \gamma \in \Gamma. \tag{9.18}$$

Example 9.4: Let ξ be an uncertain set on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ and let A be a crisp set. Then $\xi + A$ is also an uncertain set determined by

$$(\xi + A)(\gamma) = \xi(\gamma) + A, \quad \forall \gamma \in \Gamma. \tag{9.19}$$

Example 9.5: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2, \gamma_3\}$. Let ξ and η be two uncertain sets,

$$\xi(\gamma) = \begin{cases} [1,2], & \text{if } \gamma = \gamma_1 \\ [1,3], & \text{if } \gamma = \gamma_2 \\ [1,4], & \text{if } \gamma = \gamma_3, \end{cases} \qquad \eta(\gamma) = \begin{cases} (2,3), & \text{if } \gamma = \gamma_1 \\ (2,4), & \text{if } \gamma = \gamma_2 \\ (2,5), & \text{if } \gamma = \gamma_3. \end{cases}$$

Then their sum is

$$(\xi + \eta)(\gamma) = \begin{cases} (3,5), & \text{if } \gamma = \gamma_1 \\ (3,7), & \text{if } \gamma = \gamma_2 \\ (3,9), & \text{if } \gamma = \gamma_3, \end{cases}$$

and their product is

$$(\xi \times \eta)(\gamma) = \begin{cases} (2,6), & \text{if } \gamma = \gamma_1 \\ (2,12), & \text{if } \gamma = \gamma_2 \\ (2,20), & \text{if } \gamma = \gamma_3. \end{cases}$$

9.2 Membership Function

Definition 9.4 *(Liu [133]) An uncertain set ξ is said to have a membership function μ if for any Borel set B, we have*

$$\mathcal{M}\{B \subset \xi\} = \inf_{x \in B} \mu(x), \tag{9.20}$$

$$\mathcal{M}\{\xi \subset B\} = 1 - \sup_{x \in B^c} \mu(x). \tag{9.21}$$

The above equations will be called measure inversion formulas.

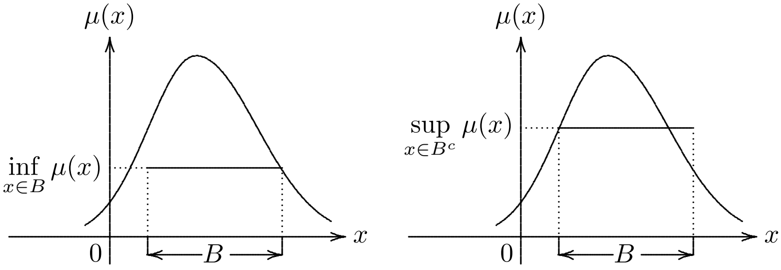


Figure 9.2: $\mathcal{M}\{B \subset \xi\} = \inf_{x\in B}\mu(x)$ and $\mathcal{M}\{\xi \subset B\} = 1 - \sup_{x\in B^c}\mu(x)$. Reprinted from Liu [133].

Remark 9.3: When an uncertain set ξ does have a membership function μ, it follows from the first measure inversion formula that

$$\mu(x) = \mathcal{M}\{x \in \xi\}. \tag{9.22}$$

Remark 9.4: The value of $\mu(x)$ represents the membership degree that x belongs to the uncertain set ξ. If $\mu(x) = 1$, then x completely belongs to ξ; if $\mu(x) = 0$, then x does not belong to ξ at all. Thus the larger the value of $\mu(x)$ is, the more true x belongs to ξ.

Remark 9.5: If an element x belongs to an uncertain set with membership degree α, then x does not belong to the uncertain set with membership degree $1-\alpha$. This fact follows from the duality property of uncertain measure. In other words, if the uncertain set has a membership function μ, then for any real number x, we have $\mathcal{M}\{x \notin \xi\} = 1 - \mathcal{M}\{x \in \xi\} = 1 - \mu(x)$. That is,

$$\mathcal{M}\{x \notin \xi\} = 1 - \mu(x). \tag{9.23}$$

Exercise 9.1: The set $\Re$ of real numbers is a special uncertain set $\xi(\gamma) \equiv \Re$. Show that such an uncertain set has a membership function

$$\mu(x) \equiv 1, \quad \forall x \in \Re \tag{9.24}$$

that is just the characteristic function of $\Re$.

Exercise 9.2: The empty set $\emptyset$ is a special uncertain set $\xi(\gamma) \equiv \emptyset$. Show that such an uncertain set has a membership function

$$\mu(x) \equiv 0, \quad \forall x \in \Re \tag{9.25}$$

that is just the characteristic function of $\emptyset$.

Exercise 9.3: A crisp set A of real numbers is a special uncertain set $\xi(\gamma) \equiv A$. Show that such an uncertain set has a membership function

$$\mu(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \tag{9.26}$$

that is just the characteristic function of A.

Exercise 9.4: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be the interval $[0, 1]$ with Borel algebra and Lebesgue measure. (i) Show that the uncertain set

$$\xi(\gamma) = [\gamma - 1,\ 1 - \gamma] \tag{9.27}$$

has a membership function

$$\mu(x) = \begin{cases} 1 - |x|, & \text{if } x \in [-1, 1] \\ 0, & \text{otherwise.} \end{cases} \tag{9.28}$$

(ii) What is the membership function of $\xi(\gamma) = (\gamma - 1,\ 1 - \gamma)$? (iii) What do those two uncertain sets make you think about?

Exercise 9.5: It is not true that every uncertain set has a membership function. Show that the uncertain set

$$\xi = \begin{cases} [2, 4] \text{ with uncertain measure } 0.6 \\ [1, 3] \text{ with uncertain measure } 0.4 \end{cases} \tag{9.29}$$

has no membership function.

Definition 9.5 *An uncertain set ξ is called triangular if it has a membership function*

$$\mu(x) = \begin{cases} \dfrac{x - a}{b - a}, & \text{if } a \le x \le b \\ \dfrac{x - c}{b - c}, & \text{if } b \le x \le c \end{cases} \tag{9.30}$$

denoted by (a, b, c) where a, b, c are real numbers with $a < b < c$.

Definition 9.6 *An uncertain set ξ is called trapezoidal if it has a membership function*

$$\mu(x) = \begin{cases} \dfrac{x - a}{b - a}, & \text{if } a \le x \le b \\ 1, & \text{if } b \le x \le c \\ \dfrac{x - d}{c - d}, & \text{if } c \le x \le d \end{cases} \tag{9.31}$$

denoted by (a, b, c, d) where a, b, c, d are real numbers with $a < b < c < d$.

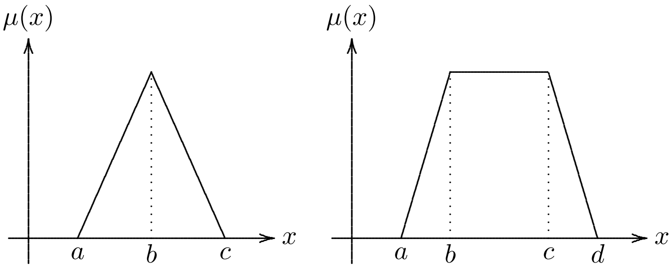


Figure 9.3: Triangular and Trapezoidal Membership Functions. Reprinted from Liu [133].

What is "young"?

Sometimes we say "those students are young". What ages can be considered "young"? In this case, "young" may be regarded as an uncertain set whose membership function is

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/5, & \text{if } 15 \le x \le 20 \\ 1, & \text{if } 20 \le x \le 35 \\ (45-x)/10, & \text{if } 35 \le x \le 45 \\ 0, & \text{if } x \ge 45. \end{cases} \tag{9.32}$$

Note that we do not say "young" if the age is below 15.

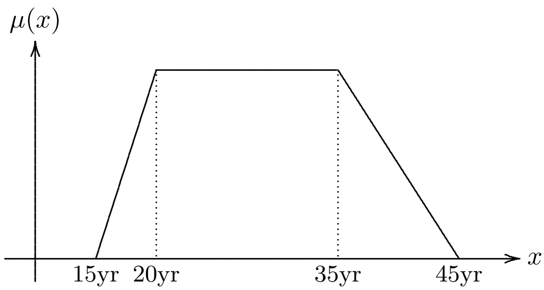


Figure 9.4: Membership Function of "young"

What is "tall"?

Sometimes we say "those sportsmen are tall". What heights (centimeters) can be considered "tall"? In this case, "tall" may be regarded as an uncertain set whose membership function is

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 180 \\ (x-180)/5, & \text{if } 180 \le x \le 185 \\ 1, & \text{if } 185 \le x \le 195 \\ (200-x)/5, & \text{if } 195 \le x \le 200 \\ 0, & \text{if } x \ge 200. \end{cases} \tag{9.33}$$

Note that we do not say "tall" if the height is over 200cm.

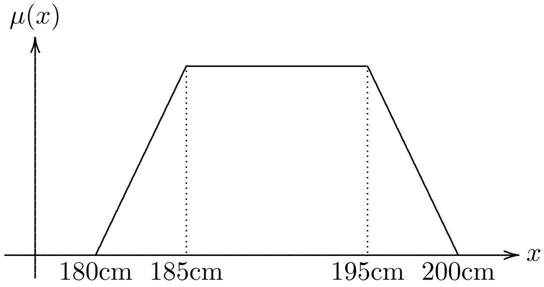


Figure 9.5: Membership Function of "tall"

What is "warm"?

Sometimes we say "those days are warm". What temperatures can be considered "warm"? In this case, "warm" may be regarded as an uncertain set whose membership function is

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/3, & \text{if } 15 \le x \le 18 \\ 1, & \text{if } 18 \le x \le 24 \\ (28-x)/4, & \text{if } 24 \le x \le 28 \\ 0, & \text{if } 28 \le x. \end{cases} \tag{9.34}$$

What is "most"?

Sometimes we say "most students are boys". What percentages can be considered "most"? In this case, "most" may be regarded as an uncertain set

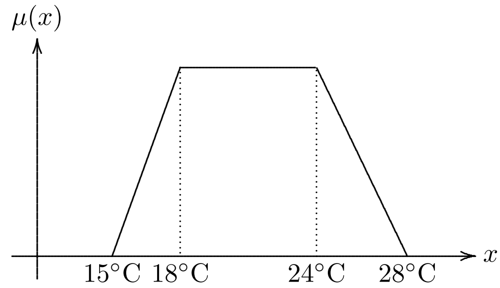


Figure 9.6: Membership Function of “warm”

whose membership function is

$$\mu(x)=\begin{cases}0, & \text{if } 0\le x\le 0.7\\ 20(x-0.7), & \text{if } 0.7\le x\le 0.75\\ 1, & \text{if } 0.75\le x\le 0.85\\ 20(0.9-x), & \text{if } 0.85\le x\le 0.9\\ 0, & \text{if } 0.9\le x\le 1.\end{cases} \tag{9.35}$$

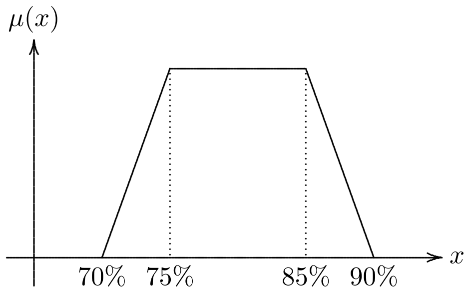


Figure 9.7: Membership Function of “most”

What uncertain sets have membership functions?

It is known that some uncertain sets do not have membership functions. What uncertain sets have membership functions?

Case I: If an uncertain set ξ degenerates to a crisp set A, then ξ has a membership function that is just the characteristic function of A.

Case II: Let ξ be an uncertain set taking values in a nested class of sets. That is, for any given γ_1 and $\gamma_2\in\Gamma$, at least one of the following alternatives holds,

$$(i)\quad \xi(\gamma_1)\subset\xi(\gamma_2), \tag{9.36}$$

$$(ii) \quad \xi(\gamma_2) \subset \xi(\gamma_1). \tag{9.37}$$

Then the uncertain set ξ has a membership function.

Sufficient and Necessary Condition

Theorem 9.13 *(Liu [130]) A real-valued function μ is a membership function if and only if*

$$0 \le \mu(x) \le 1. \tag{9.38}$$

Proof: If μ is a membership function of some uncertain set ξ, then $\mu(x) = \mathcal{M}\{x \in \xi\}$ and $0 \le \mu(x) \le 1$. Conversely, suppose μ is a function such that $0 \le \mu(x) \le 1$. Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be the interval $[0, 1]$ with Borel algebra and Lebesgue measure. Then the uncertain set

$$\xi(\gamma) = \{x \in \Re \,|\, \mu(x) \ge \gamma\} \tag{9.39}$$

has the membership function μ. See Figure 9.8.

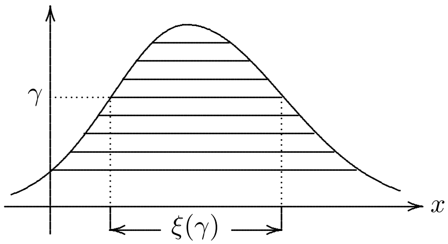


Figure 9.8: Take $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. Then $\xi(\gamma) = \{x \in \Re \,|\, \mu(x) \ge \gamma\}$ has the membership function μ. Keep in mind that ξ is not the unique uncertain set whose membership function is μ.

Membership Function of Nonempty Uncertain Set

An uncertain set ξ is said to be nonempty if $\xi(\gamma) \ne \emptyset$ for almost all $\gamma \in \Gamma$. That is,

$$\mathcal{M}\{\xi = \emptyset\} = 0. \tag{9.40}$$

Note that nonempty uncertain set does not necessarily have a membership function. However, when it does have, the following theorem gives a sufficient and necessary condition of membership function.

Theorem 9.14 *Let ξ be an uncertain set whose membership function μ exists. Then ξ is nonempty if and only if*

$$\sup_{x \in \Re} \mu(x) = 1. \tag{9.41}$$

Proof: Since the membership function μ exists, it follows from the measure inversion formula that

$$\mathcal{M}\{\xi = \emptyset\} = 1 - \sup_{x \in \emptyset^c} \mu(x) = 1 - \sup_{x \in \Re} \mu(x).$$

Thus ξ is a nonempty uncertain set if and only if (9.41) holds.

Inverse Membership Function

Definition 9.7 *(Liu [133]) Let ξ be an uncertain set with membership function μ. Then the set-valued function*

$$\mu^{-1}(\alpha) = \left\{x \in \Re \;\middle|\; \mu(x) \ge \alpha\right\}, \quad \forall \alpha \in [0,1] \tag{9.42}$$

is called the inverse membership function of ξ. Sometimes, for each given α, the set $\mu^{-1}(\alpha)$ is also called the α-cut of μ.

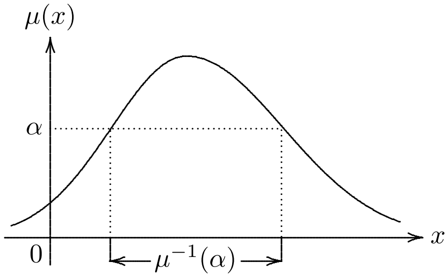


Figure 9.9: Inverse Membership Function $\mu^{-1}(\alpha)$. Reprinted from Liu [133].

Remark 9.6: It is clear that inverse membership function always exists. Please also note that $\mu^{-1}(\alpha)$ may take value of the empty set $\emptyset$.

Example 9.6: The triangular uncertain set $\xi = (a,b,c)$ has an inverse membership function

$$\mu^{-1}(\alpha) = [(1-\alpha)a + \alpha b, \alpha b + (1-\alpha)c]. \tag{9.43}$$

Example 9.7: The trapezoidal uncertain set $\xi = (a,b,c,d)$ has an inverse membership function

$$\mu^{-1}(\alpha) = [(1-\alpha)a + \alpha b, \alpha c + (1-\alpha)d]. \tag{9.44}$$

Theorem 9.15 *Let ξ be an uncertain set with inverse membership function $\mu^{-1}(\alpha)$. Then the membership function of ξ is determined by*

$$\mu(x) = \sup\left\{\alpha \in [0,1] \;\middle|\; x \in \mu^{-1}(\alpha)\right\}. \tag{9.45}$$

Proof: It is easy to verify that μ^{-1} is the inverse membership function of μ. Thus μ is the membership function of ξ.

Theorem 9.16 *(Liu [133], Sufficient and Necessary Condition) A function $\mu^{-1}(\alpha)$ is an inverse membership function if and only if it is a monotone decreasing set-valued function with respect to $\alpha \in [0,1]$. That is,*

$$\mu^{-1}(\alpha) \subset \mu^{-1}(\beta), \quad \text{if } \alpha > \beta. \tag{9.46}$$

Proof: Suppose $\mu^{-1}(\alpha)$ is an inverse membership function of some uncertain set. For any $x \in \mu^{-1}(\alpha)$, we have $\mu(x) \geq \alpha$. Since $\alpha > \beta$, we have $\mu(x) > \beta$ and then $x \in \mu^{-1}(\beta)$. Hence $\mu^{-1}(\alpha) \subset \mu^{-1}(\beta)$. Conversely, suppose $\mu^{-1}(\alpha)$ is a monotone decreasing set-valued function. Then

$$\mu(x) = \sup\left\{\alpha \in [0,1] \;\middle|\; x \in \mu^{-1}(\alpha)\right\}$$

is a membership function of some uncertain set. It is easy to verify that $\mu^{-1}(\alpha)$ is the inverse membership function of the uncertain set. The theorem is proved.

Uncertain set does not necessarily take values of its α-cuts!

Please keep in mind that uncertain set does not necessarily take values of its α-cuts. In fact, an α-cut is included in the uncertain set with uncertain measure α. Conversely, the uncertain set is included in its α-cut with uncertain measure $1-\alpha$. More precisely, we have the following theorem.

Theorem 9.17 *(Liu [133]) Let ξ be an uncertain set with inverse membership function $\mu^{-1}(\alpha)$. Then for each $\alpha \in [0,1]$, we have*

$$\mathcal{M}\{\mu^{-1}(\alpha) \subset \xi\} \geq \alpha, \tag{9.47}$$

$$\mathcal{M}\{\xi \subset \mu^{-1}(\alpha)\} \geq 1-\alpha. \tag{9.48}$$

Proof: For each $x \in \mu^{-1}(\alpha)$, we have $\mu(x) \geq \alpha$. It follows from the measure inversion formula that

$$\mathcal{M}\{\mu^{-1}(\alpha) \subset \xi\} = \inf_{x \in \mu^{-1}(\alpha)} \mu(x) \geq \alpha.$$

For each $x \notin \mu^{-1}(\alpha)$, we have $\mu(x) < \alpha$. It follows from the measure inversion formula that

$$\mathcal{M}\{\xi \subset \mu^{-1}(\alpha)\} = 1 - \sup_{x \notin \mu^{-1}(\alpha)} \mu(x) \geq 1-\alpha.$$

Regular Membership Function

Definition 9.8 *(Liu [133]) A membership function μ is said to be regular if there exists a point x_0 such that $\mu(x_0) = 1$ and $\mu(x)$ is unimodal about the mode x_0. That is, $\mu(x)$ is increasing on $(-\infty, x_0]$ and decreasing on $[x_0, +\infty)$.*

If μ is a regular membership function, then $\mu^{-1}(\alpha)$ is an interval for each α. In this case, the function

$$\mu_l^{-1}(\alpha) = \inf \mu^{-1}(\alpha) \tag{9.49}$$

is called the *left inverse membership function*, and the function

$$\mu_r^{-1}(\alpha) = \sup \mu^{-1}(\alpha) \tag{9.50}$$

is called the *right inverse membership function*. It is clear that the left inverse membership function $\mu_l^{-1}(\alpha)$ is increasing, and the right inverse membership function $\mu_r^{-1}(\alpha)$ is decreasing with respect to α.

Conversely, suppose an uncertain set ξ has a left inverse membership function $\mu_l^{-1}(\alpha)$ and right inverse membership function $\mu_r^{-1}(\alpha)$. Then the membership function μ is determined by

$$\mu(x) = \begin{cases} 0, & \text{if } x \le \mu_l^{-1}(0) \\ \alpha, & \text{if } \mu_l^{-1}(0) \le x \le \mu_l^{-1}(1) \text{ and } \mu_l^{-1}(\alpha) = x \\ 1, & \text{if } \mu_l^{-1}(1) \le x \le \mu_r^{-1}(1) \\ \beta, & \text{if } \mu_r^{-1}(1) \le x \le \mu_r^{-1}(0) \text{ and } \mu_r^{-1}(\beta) = x \\ 0, & \text{if } x \ge \mu_r^{-1}(0). \end{cases} \tag{9.51}$$

Note that the values of α and β may not be unique. In this case, we will take the maximum values.

9.3 Independence

Definition 9.9 *(Liu [136]) The uncertain sets $\xi_1, \xi_2, \cdots, \xi_n$ are said to be independent if for any Borel sets $B_1, B_2, \cdots, B_n$, we have*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i^* \subset B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i^* \subset B_i\} \tag{9.52}$$

and

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i^* \subset B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i^* \subset B_i\} \tag{9.53}$$

where ξ_i^* are arbitrarily chosen from $\{\xi_i, \xi_i^c\}$, $i = 1, 2, \cdots, n$, respectively.

Remark 9.7: Note that (9.52) represents 2^n equations. For example, when $n = 2$, the four equations are

$$\mathcal{M}\{(\xi_1 \subset B_1) \cap (\xi_2 \subset B_2)\} = \mathcal{M}\{\xi_1 \subset B_1\} \wedge \mathcal{M}\{\xi_2 \subset B_2\},$$
$$\mathcal{M}\{(\xi_1^c \subset B_1) \cap (\xi_2 \subset B_2)\} = \mathcal{M}\{\xi_1^c \subset B_1\} \wedge \mathcal{M}\{\xi_2 \subset B_2\},$$
$$\mathcal{M}\{(\xi_1 \subset B_1) \cap (\xi_2^c \subset B_2)\} = \mathcal{M}\{\xi_1 \subset B_1\} \wedge \mathcal{M}\{\xi_2^c \subset B_2\},$$
$$\mathcal{M}\{(\xi_1^c \subset B_1) \cap (\xi_2^c \subset B_2)\} = \mathcal{M}\{\xi_1^c \subset B_1\} \wedge \mathcal{M}\{\xi_2^c \subset B_2\}.$$

Also note that (9.53) represents other 2^n equations. For example, when $n = 2$, the four equations are

$$\mathcal{M}\{(\xi_1 \subset B_1) \cup (\xi_2 \subset B_2)\} = \mathcal{M}\{\xi_1 \subset B_1\} \vee \mathcal{M}\{\xi_2 \subset B_2\},$$
$$\mathcal{M}\{(\xi_1^c \subset B_1) \cup (\xi_2 \subset B_2)\} = \mathcal{M}\{\xi_1^c \subset B_1\} \vee \mathcal{M}\{\xi_2 \subset B_2\},$$
$$\mathcal{M}\{(\xi_1 \subset B_1) \cup (\xi_2^c \subset B_2)\} = \mathcal{M}\{\xi_1 \subset B_1\} \vee \mathcal{M}\{\xi_2^c \subset B_2\},$$
$$\mathcal{M}\{(\xi_1^c \subset B_1) \cup (\xi_2^c \subset B_2)\} = \mathcal{M}\{\xi_1^c \subset B_1\} \vee \mathcal{M}\{\xi_2^c \subset B_2\}.$$

Theorem 9.18 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain sets, and let ξ_i^* be arbitrarily chosen uncertain sets from $\{\xi_i, \xi_i^c\}$, $i = 1, 2, \cdots, n$, respectively. Then $\xi_1, \xi_2, \cdots, \xi_n$ are independent if and only if $\xi_1^*, \xi_2^*, \cdots, \xi_n^*$ are independent.*

Proof: Let ξ_i^{**} be arbitrarily chosen uncertain sets from $\{\xi_i^*, \xi_i^{*c}\}$, $i = 1, 2, \cdots, n$, respectively. Then $\xi_1^*, \xi_2^*, \cdots, \xi_n^*$ and $\xi_1^{**}, \xi_2^{**}, \cdots, \xi_n^{**}$ represent the same 2^n combinations. This fact implies that (9.52) and (9.53) are equivalent to

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i^{**} \subset B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\left\{\xi_i^{**} \subset B_i\right\}, \tag{9.54}$$

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i^{**} \subset B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\left\{\xi_i^{**} \subset B_i\right\}. \tag{9.55}$$

Hence $\xi_1, \xi_2, \cdots, \xi_n$ are independent if and only if $\xi_1^*, \xi_2^*, \cdots, \xi_n^*$ are independent.

Exercise 9.6: Show that the following four statements are equivalent: (i) ξ_1 and ξ_2 are independent; (ii) ξ_1^c and ξ_2 are independent; (iii) ξ_1 and ξ_2^c are independent; and (iv) ξ_1^c and ξ_2^c are independent.

Theorem 9.19 *The uncertain sets $\xi_1, \xi_2, \cdots, \xi_n$ are independent if and only if for any Borel sets $B_1, B_2, \cdots, B_n$, we have*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i^* \not\subset B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\left\{\xi_i^* \not\subset B_i\right\} \tag{9.56}$$

and

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i^* \not\subset B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\left\{\xi_i^* \not\subset B_i\right\} \tag{9.57}$$

where ξ_i^* are arbitrarily chosen from $\{\xi_i, \xi_i^c\}$, $i = 1, 2, \cdots, n$, respectively.

Proof: Since $\{\xi_i^* \not\subset B_i\}^c = \{\xi_i^* \subset B_i\}$ for $i = 1, 2, \cdots, n$, it follows from the duality of uncertain measure that

$$\mathcal{M}\left\{\bigcap_{i=1}^n (\xi_i^* \not\subset B_i)\right\} = 1 - \mathcal{M}\left\{\bigcup_{i=1}^n (\xi_i^* \subset B_i)\right\}, \tag{9.58}$$

$$\bigwedge_{i=1}^n \mathcal{M}\{\xi_i^* \not\subset B_i\} = 1 - \bigvee_{i=1}^n \mathcal{M}\{\xi_i^* \subset B_i\}, \tag{9.59}$$

$$\mathcal{M}\left\{\bigcup_{i=1}^n (\xi_i^* \not\subset B_i)\right\} = 1 - \mathcal{M}\left\{\bigcap_{i=1}^n (\xi_i^* \subset B_i)\right\}, \tag{9.60}$$

$$\bigvee_{i=1}^n \mathcal{M}\{\xi_i^* \not\subset B_i\} = 1 - \bigwedge_{i=1}^n \mathcal{M}\{\xi_i^* \subset B_i\}. \tag{9.61}$$

It follows from (9.58), (9.59), (9.60) and (9.61) that (9.56) and (9.57) are valid if and only if

$$\mathcal{M}\left\{\bigcap_{i=1}^n (\xi_i^* \subset B_i)\right\} = \bigwedge_{i=1}^n \mathcal{M}\{\xi_i^* \subset B_i\}, \tag{9.62}$$

$$\mathcal{M}\left\{\bigcup_{i=1}^n (\xi_i^* \subset B_i)\right\} = \bigvee_{i=1}^n \mathcal{M}\{\xi_i^* \subset B_i\}. \tag{9.63}$$

The above two equations are also equivalent to the independence of the uncertain sets $\xi_1, \xi_2, \cdots, \xi_n$. The theorem is thus proved.

Theorem 9.20 *The uncertain sets $\xi_1, \xi_2, \cdots, \xi_n$ are independent if and only if for any Borel sets $B_1, B_2, \cdots, B_n$, we have*

$$\mathcal{M}\left\{\bigcap_{i=1}^n (B_i \subset \xi_i^*)\right\} = \bigwedge_{i=1}^n \mathcal{M}\{B_i \subset \xi_i^*\} \tag{9.64}$$

and

$$\mathcal{M}\left\{\bigcup_{i=1}^n (B_i \subset \xi_i^*)\right\} = \bigvee_{i=1}^n \mathcal{M}\{B_i \subset \xi_i^*\} \tag{9.65}$$

where ξ_i^* are arbitrarily chosen from $\{\xi_i, \xi_i^c\}$, $i = 1, 2, \cdots, n$, respectively.

Proof: Since $\{B_i \subset \xi_i^*\} = \{\xi_i^{*c} \subset B_i^c\}$ for $i = 1, 2, \cdots, n$, we immediately have

$$\mathcal{M}\left\{\bigcap_{i=1}^n (B_i \subset \xi_i^*)\right\} = \mathcal{M}\left\{\bigcap_{i=1}^n (\xi_i^{*c} \subset B_i^c)\right\}, \tag{9.66}$$

$$\bigwedge_{i=1}^n \mathcal{M}\{B_i \subset \xi_i^*\} = \bigwedge_{i=1}^n \mathcal{M}\{\xi_i^{*c} \subset B_i^c\}, \tag{9.67}$$

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(B_i \subset \xi_i^*)\right\} = \mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i^{*c} \subset B_i^c)\right\}, \tag{9.68}$$

$$\bigvee_{i=1}^{n} \mathcal{M}\{B_i \subset \xi_i^*\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i^{*c} \subset B_i^c\}. \tag{9.69}$$

It follows from (9.66), (9.67), (9.68) and (9.69) that (9.64) and (9.65) are valid if and only if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i^{*c} \subset B_i^c)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i^{*c} \subset B_i^c\}, \tag{9.70}$$

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i^{*c} \subset B_i^c)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i^{*c} \subset B_i^c\}. \tag{9.71}$$

The above two equations are also equivalent to the independence of the uncertain sets $\xi_1, \xi_2, \cdots, \xi_n$. The theorem is thus proved.

Theorem 9.21 *The uncertain sets $\xi_1, \xi_2, \cdots, \xi_n$ are independent if and only if for any Borel sets $B_1, B_2, \cdots, B_n$, we have*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(B_i \not\subset \xi_i^*)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{B_i \not\subset \xi_i^*\} \tag{9.72}$$

and

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(B_i \not\subset \xi_i^*)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{B_i \not\subset \xi_i^*\} \tag{9.73}$$

where ξ_i^* are arbitrarily chosen from $\{\xi_i, \xi_i^c\}$, $i = 1, 2, \cdots, n$, respectively.

Proof: Since $\{B_i \not\subset \xi_i^*\}^c = \{B_i \subset \xi_i^*\}$ for $i = 1, 2, \cdots, n$, it follows from the duality of uncertain measure that

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(B_i \not\subset \xi_i^*)\right\} = 1 - \mathcal{M}\left\{\bigcup_{i=1}^{n}(B_i \subset \xi_i^*)\right\}, \tag{9.74}$$

$$\bigwedge_{i=1}^{n} \mathcal{M}\{B_i \not\subset \xi_i^*\} = 1 - \bigvee_{i=1}^{n} \mathcal{M}\{B_i \subset \xi_i^*\}, \tag{9.75}$$

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(B_i \not\subset \xi_i^*)\right\} = 1 - \mathcal{M}\left\{\bigcap_{i=1}^{n}(B_i \subset \xi_i^*)\right\}, \tag{9.76}$$

$$\bigvee_{i=1}^{n} \mathcal{M}\{B_i \not\subset \xi_i^*\} = 1 - \bigwedge_{i=1}^{n} \mathcal{M}\{B_i \subset \xi_i^*\}. \tag{9.77}$$

It follows from (9.74), (9.75), (9.76) and (9.77) that (9.72) and (9.73) are valid if and only if

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(B_i \subset \xi_i^*)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{B_i \subset \xi_i^*\}, \tag{9.78}$$

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(B_i \subset \xi_i^*)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{B_i \subset \xi_i^*\}. \tag{9.79}$$

The above two equations are also equivalent to the independence of the uncertain sets $\xi_1, \xi_2, \cdots, \xi_n$. The theorem is thus proved.

9.4 Set Operational Law

This section will discuss the union, intersection and complement of independent uncertain sets via membership functions.

Union of Uncertain Sets

Theorem 9.22 *(Liu [133]) Let ξ and η be independent uncertain sets with membership functions μ and ν, respectively. Then their union $\xi \cup \eta$ has a membership function*

$$\lambda(x) = \mu(x) \vee \nu(x). \tag{9.80}$$

Proof: In order to prove $\mu \vee \nu$ is a membership function of $\xi \cup \eta$, we must verify the two measure inversion formulas. Let B be any Borel set, and write

$$\beta = \inf_{x \in B} \mu(x) \vee \nu(x).$$

Then $B \subset \mu^{-1}(\beta) \cup \nu^{-1}(\beta)$. By the independence of ξ and η, we have

$$\begin{aligned}
\mathcal{M}\{B \subset (\xi \cup \eta)\} &\geq \mathcal{M}\{(\mu^{-1}(\beta) \cup \nu^{-1}(\beta)) \subset (\xi \cup \eta)\} \\
&\geq \mathcal{M}\{(\mu^{-1}(\beta) \subset \xi) \cap (\nu^{-1}(\beta) \subset \eta)\} \\
&= \mathcal{M}\{\mu^{-1}(\beta) \subset \xi\} \wedge \mathcal{M}\{\nu^{-1}(\beta) \subset \eta\} \\
&\geq \beta \wedge \beta = \beta.
\end{aligned}$$

Thus

$$\mathcal{M}\{B \subset (\xi \cup \eta)\} \geq \inf_{x \in B} \mu(x) \vee \nu(x). \tag{9.81}$$

On the other hand, for any $x \in B$, we have

$$\begin{aligned}
\mathcal{M}\{B \subset (\xi \cup \eta)\} &\leq \mathcal{M}\{x \in (\xi \cup \eta)\} = \mathcal{M}\{(x \in \xi) \cup (x \in \eta)\} \\
&= \mathcal{M}\{x \in \xi\} \vee \mathcal{M}\{x \in \eta\} = \mu(x) \vee \nu(x).
\end{aligned}$$

Thus

$$\mathcal{M}\{B \subset (\xi \cup \eta)\} \le \inf_{x \in B} \mu(x) \vee \nu(x). \tag{9.82}$$

It follows from (9.81) and (9.82) that

$$\mathcal{M}\{B \subset (\xi \cup \eta)\} = \inf_{x \in B} \mu(x) \vee \nu(x). \tag{9.83}$$

The first measure inversion formula is verified. Next we prove the second measure inversion formula. By the independence of ξ and η, we have

$$\begin{aligned}\mathcal{M}\{(\xi \cup \eta) \subset B\} &= \mathcal{M}\{(\xi \subset B) \cap (\eta \subset B)\} = \mathcal{M}\{\xi \subset B\} \wedge \mathcal{M}\{\eta \subset B\} \\ &= \left(1 - \sup_{x \in B^c} \mu(x)\right) \wedge \left(1 - \sup_{x \in B^c} \nu(x)\right) \\ &= 1 - \sup_{x \in B^c} \mu(x) \vee \nu(x).\end{aligned}$$

That is,

$$\mathcal{M}\{(\xi \cup \eta) \subset B\} = 1 - \sup_{x \in B^c} \mu(x) \vee \nu(x). \tag{9.84}$$

The second measure inversion formula is verified. Therefore, the union $\xi \cup \eta$ is proved to have the membership function $\mu \vee \nu$ by the measure inversion formulas (9.83) and (9.84).

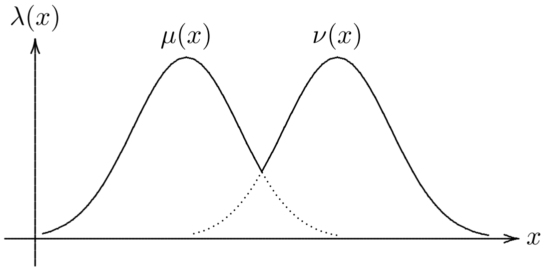


Figure 9.10: Membership Function of Union of Uncertain Sets. Reprinted from Liu [133].

Intersection of Uncertain Sets

Theorem 9.23 *(Liu [133]) Let ξ and η be independent uncertain sets with membership functions μ and ν, respectively. Then their intersection $\xi \cap \eta$ has a membership function*

$$\lambda(x) = \mu(x) \wedge \nu(x). \tag{9.85}$$

Proof: In order to prove $\mu \wedge \nu$ is a membership function of $\xi \cap \eta$, we must verify the two measure inversion formulas. Let B be any Borel set. By the independence of ξ and η, we have

$$\begin{aligned}\mathcal{M}\{B \subset (\xi \cap \eta)\} &= \mathcal{M}\{(B \subset \xi) \cap (B \subset \eta)\} = \mathcal{M}\{B \subset \xi\} \wedge \mathcal{M}\{B \subset \eta\} \\ &= \inf_{x \in B} \mu(x) \wedge \inf_{x \in B} \nu(x) = \inf_{x \in B} \mu(x) \wedge \nu(x).\end{aligned}$$

That is,

$$\mathcal{M}\{B \subset (\xi \cap \eta)\} = \inf_{x \in B} \mu(x) \wedge \nu(x). \tag{9.86}$$

The first measure inversion formula is verified. In order to prove the second measure inversion formula, we write

$$\beta = \sup_{x \in B^c} \mu(x) \wedge \nu(x).$$

Then for any given number $\varepsilon > 0$, we have $\mu^{-1}(\beta + \varepsilon) \cap \nu^{-1}(\beta + \varepsilon) \subset B$. By the independence of ξ and η, we obtain

$$\begin{aligned}\mathcal{M}\{(\xi \cap \eta) \subset B\} &\geq \mathcal{M}\{(\xi \cap \eta) \subset (\mu^{-1}(\beta + \varepsilon) \cap \nu^{-1}(\beta + \varepsilon))\} \\ &\geq \mathcal{M}\{(\xi \subset \mu^{-1}(\beta + \varepsilon)) \cap (\eta \subset \nu^{-1}(\beta + \varepsilon))\} \\ &= \mathcal{M}\{\xi \subset \mu^{-1}(\beta + \varepsilon)\} \wedge \mathcal{M}\{\eta \subset \nu^{-1}(\beta + \varepsilon)\} \\ &\geq (1 - \beta - \varepsilon) \wedge (1 - \beta - \varepsilon) = 1 - \beta - \varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$\mathcal{M}\{(\xi \cap \eta) \subset B\} \geq 1 - \sup_{x \in B^c} \mu(x) \wedge \nu(x). \tag{9.87}$$

On the other hand, for any $x \in B^c$, we have

$$\begin{aligned}\mathcal{M}\{(\xi \cap \eta) \subset B\} &\leq \mathcal{M}\{x \notin (\xi \cap \eta)\} = \mathcal{M}\{(x \notin \xi) \cup (x \notin \eta)\} \\ &= \mathcal{M}\{x \notin \xi\} \vee \mathcal{M}\{x \notin \eta\} = (1 - \mu(x)) \vee (1 - \nu(x)) \\ &= 1 - \mu(x) \wedge \nu(x).\end{aligned}$$

Thus

$$\mathcal{M}\{(\xi \cap \eta) \subset B\} \leq 1 - \sup_{x \in B^c} \mu(x) \wedge \nu(x). \tag{9.88}$$

It follows from (9.87) and (9.88) that

$$\mathcal{M}\{(\xi \cap \eta) \subset B\} = 1 - \sup_{x \in B^c} \mu(x) \wedge (x). \tag{9.89}$$

The second measure inversion formula is verified. Therefore, the intersection $\xi \cap \eta$ is proved to have the membership function $\mu \wedge \nu$ by the measure inversion formulas (9.86) and (9.89).

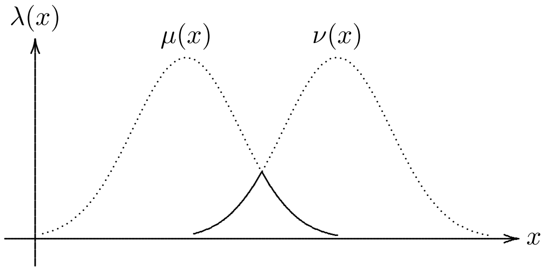


Figure 9.11: Membership Function of Intersection of Uncertain Sets. Reprinted from Liu [133].

Complement of Uncertain Set

Theorem 9.24 *(Liu [133]) Let ξ be an uncertain set with membership function μ. Then its complement ξ^c has a membership function*

$$\lambda(x) = 1 - \mu(x). \tag{9.90}$$

Proof: In order to prove $1 - \mu$ is a membership function of ξ^c, we must verify the two measure inversion formulas. Let B be a Borel set. It follows from the definition of membership function that

$$\mathcal{M}\{B \subset \xi^c\} = \mathcal{M}\{\xi \subset B^c\} = 1 - \sup_{x \in (B^c)^c} \mu(x) = \inf_{x \in B}(1 - \mu(x)),$$

$$\mathcal{M}\{\xi^c \subset B\} = \mathcal{M}\{B^c \subset \xi\} = \inf_{x \in B^c} \mu(x) = 1 - \sup_{x \in B^c}(1 - \mu(x)).$$

Thus ξ^c has a membership function $1 - \mu$.

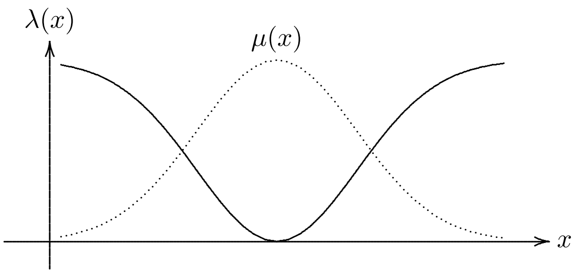


Figure 9.12: Membership Function of Complement of Uncertain Set. Reprinted from Liu [133].

9.5 Arithmetic Operational Law

This section will present an arithmetic operational law of independent uncertain sets via inverse membership functions, including addition, subtraction, multiplication and division.

Theorem 9.25 *(Liu [133]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain sets with inverse membership functions $\mu_1^{-1}, \mu_2^{-1}, \cdots, \mu_n^{-1}$, respectively. If f is a measurable function, then the uncertain set*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{9.91}$$

has an inverse membership function,

$$\lambda^{-1}(\alpha) = f(\mu_1^{-1}(\alpha), \mu_2^{-1}(\alpha), \cdots, \mu_n^{-1}(\alpha)). \tag{9.92}$$

Proof: For simplicity, we only prove the case $n = 2$. Let B be any Borel set, and write

$$\beta = \inf_{x \in B} \lambda(x).$$

Then $B \subset \lambda^{-1}(\beta)$. Since $\lambda^{-1}(\beta) = f(\mu_1^{-1}(\beta), \mu_2^{-1}(\beta))$, by the independence of ξ_1 and ξ_2, we have

$$\begin{aligned}
\mathcal{M}\{B \subset \xi\} &\geq \mathcal{M}\{\lambda^{-1}(\beta) \subset \xi\} = \mathcal{M}\{f(\mu_1^{-1}(\beta), \mu_2^{-1}(\beta)) \subset \xi\} \\
&\geq \mathcal{M}\{(\mu_1^{-1}(\beta) \subset \xi_1) \cap (\mu_2^{-1}(\beta) \subset \xi_2)\} \\
&= \mathcal{M}\{\mu_1^{-1}(\beta) \subset \xi_1\} \wedge \mathcal{M}\{\mu_2^{-1}(\beta) \subset \xi_2\} \\
&\geq \beta \wedge \beta = \beta.
\end{aligned}$$

Thus

$$\mathcal{M}\{B \subset \xi\} \geq \inf_{x \in B} \lambda(x). \tag{9.93}$$

On the other hand, for any given number $\varepsilon > 0$, we have $B \not\subset \lambda^{-1}(\beta + \varepsilon)$. Since $\lambda^{-1}(\beta + \varepsilon) = f(\mu_1^{-1}(\beta + \varepsilon), \mu_2^{-1}(\beta + \varepsilon))$, we obtain

$$\begin{aligned}
\mathcal{M}\{B \not\subset \xi\} &\geq \mathcal{M}\{\xi \subset \lambda^{-1}(\beta + \varepsilon)\} = \mathcal{M}\{\xi \subset f(\mu_1^{-1}(\beta + \varepsilon), \mu_2^{-1}(\beta + \varepsilon))\} \\
&\geq \mathcal{M}\{(\xi_1 \subset \mu_1^{-1}(\beta + \varepsilon)) \cap (\xi_2 \subset \mu_2^{-1}(\beta + \varepsilon))\} \\
&= \mathcal{M}\{\xi_1 \subset \mu_1^{-1}(\beta + \varepsilon)\} \wedge \mathcal{M}\{\xi_2 \subset \mu_2^{-1}(\beta + \varepsilon)\} \\
&\geq (1 - \beta - \varepsilon) \wedge (1 - \beta - \varepsilon) = 1 - \beta - \varepsilon
\end{aligned}$$

and then

$$\mathcal{M}\{B \subset \xi\} = 1 - \mathcal{M}\{B \not\subset \xi\} \leq \beta + \varepsilon.$$

Letting $\varepsilon \to 0$, we get

$$\mathcal{M}\{B \subset \xi\} \leq \beta = \inf_{x \in B} \lambda(x). \tag{9.94}$$

It follows from (9.93) and (9.94) that

$$\mathcal{M}\{B \subset \xi\} = \inf_{x \in B} \lambda(x). \tag{9.95}$$

The first measure inversion formula is verified. In order to prove the second measure inversion formula, we write

$$\beta = \sup_{x \in B^c} \lambda(x).$$

Then for any given number $\varepsilon > 0$, we have $\lambda^{-1}(\beta+\varepsilon) \subset B$. Please note that $\lambda^{-1}(\beta+\varepsilon) = f(\mu_1^{-1}(\beta+\varepsilon), \mu_2^{-1}(\beta+\varepsilon))$. By the independence of ξ_1 and ξ_2, we obtain

$$\begin{aligned}
\mathcal{M}\{\xi \subset B\} &\geq \mathcal{M}\{\xi \subset \lambda^{-1}(\beta+\varepsilon)\} = \mathcal{M}\{\xi \subset f(\mu_1^{-1}(\beta+\varepsilon), \mu_2^{-1}(\beta+\varepsilon))\} \\
&\geq \mathcal{M}\{(\xi_1 \subset \mu_1^{-1}(\beta+\varepsilon)) \cap (\xi_2 \subset \mu_2^{-1}(\beta+\varepsilon))\} \\
&= \mathcal{M}\{\xi_1 \subset \mu_1^{-1}(\beta+\varepsilon)\} \wedge \mathcal{M}\{\xi_2 \subset \mu_2^{-1}(\beta+\varepsilon)\} \\
&\geq (1-\beta-\varepsilon) \wedge (1-\beta-\varepsilon) = 1-\beta-\varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$\mathcal{M}\{\xi \subset B\} \geq 1 - \sup_{x \in B^c} \lambda(x). \tag{9.96}$$

On the other hand, for any given number $\varepsilon > 0$, we have $\lambda^{-1}(\beta-\varepsilon) \not\subset B$. Since $\lambda^{-1}(\beta-\varepsilon) = f(\mu_1^{-1}(\beta-\varepsilon), \mu_2^{-1}(\beta-\varepsilon))$, we obtain

$$\begin{aligned}
\mathcal{M}\{\xi \not\subset B\} &\geq \mathcal{M}\{\lambda^{-1}(\beta-\varepsilon) \subset \xi\} = \mathcal{M}\{f(\mu_1^{-1}(\beta-\varepsilon), \mu_2^{-1}(\beta-\varepsilon)) \subset \xi\} \\
&\geq \mathcal{M}\{(\mu_1^{-1}(\beta-\varepsilon) \subset \xi_1) \cap (\mu_2^{-1}(\beta-\varepsilon) \subset \xi_2)\} \\
&= \mathcal{M}\{\mu_1^{-1}(\beta-\varepsilon) \subset \xi_1\} \wedge \mathcal{M}\{\mu_2^{-1}(\beta-\varepsilon) \subset \xi_2\} \\
&\geq (\beta-\varepsilon) \wedge (\beta-\varepsilon) = \beta-\varepsilon
\end{aligned}$$

and then

$$\mathcal{M}\{\xi \subset B\} = 1 - \mathcal{M}\{\xi \not\subset B\} \leq 1-\beta+\varepsilon.$$

Letting $\varepsilon \to 0$, we get

$$\mathcal{M}\{\xi \subset B\} \leq 1-\beta = 1 - \sup_{x \in B^c} \lambda(x). \tag{9.97}$$

It follows from (9.96) and (9.97) that

$$\mathcal{M}\{\xi \subset B\} = 1 - \sup_{x \in B^c} \lambda(x). \tag{9.98}$$

The second measure inversion formula is verified. Therefore, ξ is proved to have the membership function λ by the measure inversion formulas (9.95) and (9.98).

Example 9.8: Let $\xi = (a_1, a_2, a_3)$ and $\eta = (b_1, b_2, b_3)$ be two independent triangular uncertain sets. At first, ξ has an inverse membership function,

$$\mu^{-1}(\alpha) = [(1-\alpha)a_1 + \alpha a_2, \alpha a_2 + (1-\alpha)a_3], \tag{9.99}$$

and η has an inverse membership function,

$$\nu^{-1}(\alpha) = [(1-\alpha)b_1 + \alpha b_2, \alpha b_2 + (1-\alpha)b_3]. \tag{9.100}$$

It follows from the operational law that the sum $\xi + \eta$ has an inverse membership function,

$$\lambda^{-1}(\alpha) = [(1-\alpha)(a_1+b_1)+\alpha(a_2+b_2), \alpha(a_2+b_2)+(1-\alpha)(a_3+b_3)]. \tag{9.101}$$

In other words, the sum $\xi + \eta$ is also a triangular uncertain set, and

$$\xi + \eta = (a_1 + b_1, a_2 + b_2, a_3 + b_3). \tag{9.102}$$

Example 9.9: Let $\xi = (a_1, a_2, a_3)$ and $\eta = (b_1, b_2, b_3)$ be two independent triangular uncertain sets. It follows from the operational law that the difference $\xi - \eta$ has an inverse membership function,

$$\lambda^{-1}(\alpha) = [(1-\alpha)(a_1-b_3)+\alpha(a_2-b_2), \alpha(a_2-b_2)+(1-\alpha)(a_3-b_1)]. \tag{9.103}$$

In other words, the difference $\xi - \eta$ is also a triangular uncertain set, and

$$\xi - \eta = (a_1 - b_3, a_2 - b_2, a_3 - b_1). \tag{9.104}$$

Example 9.10: Let $\xi = (a_1, a_2, a_3)$ be a triangular uncertain set, and k a real number. When $k \ge 0$, the product $k \cdot \xi$ has an inverse membership function,

$$\lambda^{-1}(\alpha) = [(1-\alpha)(ka_1) + \alpha(ka_2), \alpha(ka_2) + (1-\alpha)(ka_3)]. \tag{9.105}$$

That is, the product $k \cdot \xi$ is a triangular uncertain set (ka_1, ka_2, ka_3). When $k < 0$, the product $k \cdot \xi$ has an inverse membership function,

$$\lambda^{-1}(\alpha) = [(1-\alpha)(ka_3) + \alpha(ka_2), \alpha(ka_2) + (1-\alpha)(ka_1)]. \tag{9.106}$$

That is, the product $k \cdot \xi$ is a triangular uncertain set (ka_3, ka_2, ka_1). In summary, we have

$$k \cdot \xi = \begin{cases} (ka_1, ka_2, ka_3), & \text{if } k \ge 0 \\ (ka_3, ka_2, ka_1), & \text{if } k < 0. \end{cases} \tag{9.107}$$

Exercise 9.7: Let $\xi = (a_1, a_2, a_3, a_4)$ and $\eta = (b_1, b_2, b_3, b_4)$ be two independent trapezoidal uncertain sets, and k a real number. Show that

$$\xi + \eta = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4), \tag{9.108}$$

$$\xi - \eta = (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1), \tag{9.109}$$

$$k \cdot \xi = \begin{cases} (ka_1, ka_2, ka_3, ka_4), & \text{if } k \ge 0 \\ (ka_4, ka_3, ka_2, ka_1), & \text{if } k < 0. \end{cases} \tag{9.110}$$

Monotone Function of Regular Uncertain Sets

In practice, it is usually required to deal with monotone functions of regular uncertain sets. In this case, we have the following shortcut.

Theorem 9.26 *(Liu [133]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain sets with regular membership functions $\mu_1, \mu_2, \cdots, \mu_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then the uncertain set*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{9.111}$$

has a regular membership function, and

$$\lambda_l^{-1}(\alpha) = f(\mu_{1l}^{-1}(\alpha), \cdots, \mu_{ml}^{-1}(\alpha), \mu_{m+1,r}^{-1}(\alpha), \cdots, \mu_{nr}^{-1}(\alpha)), \tag{9.112}$$

$$\lambda_r^{-1}(\alpha) = f(\mu_{1r}^{-1}(\alpha), \cdots, \mu_{mr}^{-1}(\alpha), \mu_{m+1,l}^{-1}(\alpha), \cdots, \mu_{nl}^{-1}(\alpha)), \tag{9.113}$$

where $\lambda_l^{-1}, \mu_{1l}^{-1}, \mu_{2l}^{-1}, \cdots, \mu_{nl}^{-1}$ are left inverse membership functions, and λ_r^{-1}, $\mu_{1r}^{-1}, \mu_{2r}^{-1}, \cdots, \mu_{nr}^{-1}$ are right inverse membership functions of $\xi, \xi_1, \xi_2, \cdots, \xi_n$, respectively.

Proof: Note that $\mu_1^{-1}(\alpha), \mu_2^{-1}(\alpha), \cdots, \mu_n^{-1}(\alpha)$ are intervals for each α. Since $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, the value

$$\lambda^{-1}(\alpha) = f(\mu_1^{-1}(\alpha), \cdots, \mu_m^{-1}(\alpha), \mu_{m+1}^{-1}(\alpha), \cdots, \mu_n^{-1}(\alpha))$$

is also an interval. Thus ξ has a regular membership function, and its left and right inverse membership functions are determined by (9.112) and (9.113), respectively.

Exercise 9.8: Let ξ and η be independent uncertain sets with left inverse membership functions μ_l^{-1} and ν_l^{-1} and right inverse membership functions μ_r^{-1} and ν_r^{-1}, respectively. Show that the sum $\xi + \eta$ has left and right inverse membership functions,

$$\lambda_l^{-1}(\alpha) = \mu_l^{-1}(\alpha) + \nu_l^{-1}(\alpha), \tag{9.114}$$

$$\lambda_r^{-1}(\alpha) = \mu_r^{-1}(\alpha) + \nu_r^{-1}(\alpha). \tag{9.115}$$

Exercise 9.9: Let ξ and η be independent uncertain sets with left inverse membership functions μ_l^{-1} and ν_l^{-1} and right inverse membership functions μ_r^{-1} and ν_r^{-1}, respectively. Show that the difference $\xi - \eta$ has left and right inverse membership functions,

$$\lambda_l^{-1}(\alpha) = \mu_l^{-1}(\alpha) - \nu_r^{-1}(\alpha), \tag{9.116}$$

$$\lambda_r^{-1}(\alpha) = \mu_r^{-1}(\alpha) - \nu_l^{-1}(\alpha). \tag{9.117}$$

Exercise 9.10: Let ξ and η be independent and positive uncertain sets with left inverse membership functions μ_l^{-1} and ν_l^{-1} and right inverse membership functions μ_r^{-1} and ν_r^{-1}, respectively. Show that

$$\frac{\xi}{\xi+\eta} \tag{9.118}$$

has left and right inverse membership functions,

$$\lambda_l^{-1}(\alpha) = \frac{\mu_l^{-1}(\alpha)}{\mu_l^{-1}(\alpha) + \nu_r^{-1}(\alpha)}, \tag{9.119}$$

$$\lambda_r^{-1}(\alpha) = \frac{\mu_r^{-1}(\alpha)}{\mu_r^{-1}(\alpha) + \nu_l^{-1}(\alpha)}. \tag{9.120}$$

9.6 Expected Value

Recall that an uncertain set ξ is nonempty if $\xi(\gamma) \neq \emptyset$ for almost all $\gamma \in \Gamma$. This section will introduce a concept of expected value for nonempty uncertain set.

Definition 9.10 *(Liu [127]) Let ξ be a nonempty uncertain set. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \succeq x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \preceq x\}\mathrm{d}x \tag{9.121}$$

provided that at least one of the two integrals is finite.

Please note that $\xi \succeq x$ represents "ξ is imaginarily included in $[x, +\infty)$", and $\xi \preceq x$ represents "ξ is imaginarily included in $(-\infty, x]$". What are the appropriate values of $\mathcal{M}\{\xi \succeq x\}$ and $\mathcal{M}\{\xi \preceq x\}$? Unfortunately, this problem is not as simple as you think.

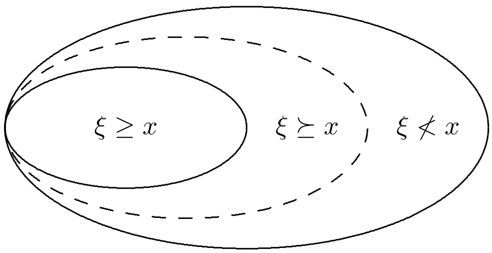


Figure 9.13: $\{\xi \geq x\} \subset \{\xi \succeq x\} \subset \{\xi \not< x\}$

Intuitively, for $\mathcal{M}\{\xi \succeq x\}$, it is too conservative if we take the value $\mathcal{M}\{\xi \geq x\}$, and it is too adventurous if we take the value $1-\mathcal{M}\{\xi < x\}$. Thus we assign $\mathcal{M}\{\xi \succeq x\}$ the middle value between $\mathcal{M}\{\xi \geq x\}$ and $1-\mathcal{M}\{\xi < x\}$. That is,

$$\mathcal{M}\{\xi \succeq x\} = \frac{1}{2}\left(\mathcal{M}\{\xi \geq x\} + 1 - \mathcal{M}\{\xi < x\}\right). \tag{9.122}$$

Similarly, we also define

$$\mathcal{M}\{\xi \preceq x\} = \frac{1}{2}\left(\mathcal{M}\{\xi \leq x\} + 1 - \mathcal{M}\{\xi > x\}\right). \tag{9.123}$$

Example 9.11: In order to illustrate the expected value operator, let us consider an uncertain set,

$$\xi = \begin{cases} [1,2] \text{ with uncertain measure } 0.6 \\ [2,3] \text{ with uncertain measure } 0.3 \\ [3,4] \text{ with uncertain measure } 0.2. \end{cases}$$

It follows from the definition of $\mathcal{M}\{\xi \succeq x\}$ and $\mathcal{M}\{\xi \preceq x\}$ that

$$\mathcal{M}\{\xi \succeq x\} = \begin{cases} 1, & \text{if } x \leq 1 \\ 0.7, & \text{if } 1 < x \leq 2 \\ 0.3, & \text{if } 2 < x \leq 3 \\ 0.1, & \text{if } 3 < x \leq 4 \\ 0, & \text{if } x > 4, \end{cases}$$

$$\mathcal{M}\{\xi \preceq x\} \equiv 0, \quad \forall x \leq 0.$$

Thus

$$E[\xi] = \int_0^1 1\mathrm{d}x + \int_1^2 0.7\mathrm{d}x + \int_2^3 0.3\mathrm{d}x + \int_3^4 0.1\mathrm{d}x = 2.1.$$

How to Obtain Expected Value from Membership Function?

Let ξ be an uncertain set with membership function μ. In order to calculate its expected value via (9.121), we must determine the values of $\mathcal{M}\{\xi \succeq x\}$ and $\mathcal{M}\{\xi \preceq x\}$ from the membership function μ.

Theorem 9.27 *Let ξ be an uncertain set with membership function μ. Then for any real number x, we have*

$$\mathcal{M}\{\xi \succeq x\} = \frac{1}{2}\left(\sup_{y \geq x} \mu(y) + 1 - \sup_{y < x} \mu(y)\right), \tag{9.124}$$

$$\mathcal{M}\{\xi \preceq x\} = \frac{1}{2}\left(\sup_{y \leq x} \mu(y) + 1 - \sup_{y > x} \mu(y)\right). \tag{9.125}$$

Proof: Since the uncertain set ξ has a membership function μ, the second measure inversion formula tells us that

$$\mathcal{M}\{\xi \ge x\} = 1 - \sup_{y<x} \mu(y),$$

$$\mathcal{M}\{\xi < x\} = 1 - \sup_{y\ge x} \mu(y).$$

Thus (9.124) follows from (9.122) immediately. We may also prove (9.125) similarly.

Theorem 9.28 *(Liu [129]) Let ξ be an uncertain set with regular membership function μ. Then*

$$E[\xi] = x_0 + \frac{1}{2}\int_{x_0}^{+\infty} \mu(x)\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{x_0} \mu(x)\mathrm{d}x \tag{9.126}$$

where x_0 is a point such that $\mu(x_0) = 1$.

Proof: Since μ is increasing on $(-\infty, x_0]$ and decreasing on $[x_0, +\infty)$, it follows from Theorem 9.27 that for almost all x, we have

$$\mathcal{M}\{\xi \succeq x\} = \begin{cases} 1 - \mu(x)/2, & \text{if } x \le x_0 \\ \mu(x)/2, & \text{if } x \ge x_0 \end{cases} \tag{9.127}$$

and

$$\mathcal{M}\{\xi \preceq x\} = \begin{cases} \mu(x)/2, & \text{if } x \le x_0 \\ 1 - \mu(x)/2, & \text{if } x \ge x_0 \end{cases} \tag{9.128}$$

for any real number x. If $x_0 \ge 0$, then

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} \mathcal{M}\{\xi \succeq x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \preceq x\}\mathrm{d}x \\ &= \int_0^{x_0} \left(1 - \frac{\mu(x)}{2}\right)\mathrm{d}x + \int_{x_0}^{+\infty} \frac{\mu(x)}{2}\mathrm{d}x - \int_{-\infty}^0 \frac{\mu(x)}{2}\mathrm{d}x \\ &= x_0 + \frac{1}{2}\int_{x_0}^{+\infty} \mu(x)\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{x_0} \mu(x)\mathrm{d}x. \end{aligned}$$

If $x_0 < 0$, then

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} \mathcal{M}\{\xi \succeq x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \preceq x\}\mathrm{d}x \\ &= \int_0^{+\infty} \frac{\mu(x)}{2}\mathrm{d}x - \int_{-\infty}^{x_0} \frac{\mu(x)}{2}\mathrm{d}x - \int_{x_0}^0 \left(1 - \frac{\mu(x)}{2}\right)\mathrm{d}x \\ &= x_0 + \frac{1}{2}\int_{x_0}^{+\infty} \mu(x)\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{x_0} \mu(x)\mathrm{d}x. \end{aligned}$$

The theorem is thus proved.

Remark 9.8: If the membership function of the uncertain set ξ is not assumed to be regular, then

$$E[\xi] = x_0 + \frac{1}{2}\int_{x_0}^{+\infty} \sup_{y \ge x} \mu(y)\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{x_0} \sup_{y \le x} \mu(y)\mathrm{d}x. \tag{9.129}$$

Exercise 9.11: Show that the triangular uncertain set $\xi = (a, b, c)$ has an expected value

$$E[\xi] = \frac{a + 2b + c}{4}. \tag{9.130}$$

Exercise 9.12: Show that the trapezoidal uncertain set $\xi = (a, b, c, d)$ has an expected value

$$E[\xi] = \frac{a + b + c + d}{4}. \tag{9.131}$$

Theorem 9.29 *(Liu [133]) Let ξ be a nonempty uncertain set with membership function μ. Then*

$$E[\xi] = \frac{1}{2}\int_0^1 \left(\inf \mu^{-1}(\alpha) + \sup \mu^{-1}(\alpha)\right) \mathrm{d}\alpha \tag{9.132}$$

where $\inf \mu^{-1}(\alpha)$ *and* $\sup \mu^{-1}(\alpha)$ *are the infimum and supremum of the α-cut, respectively.*

Proof: Since ξ is a nonempty uncertain set and has a finite expected value, we may assume that there exists a point x_0 such that $\mu(x_0) = 1$ (perhaps after a small perturbation). It is clear that the two integrals

$$\int_{x_0}^{+\infty} \sup_{y \ge x} \mu(y)\mathrm{d}x \quad \text{and} \quad \int_0^1 (\sup \mu^{-1}(\alpha) - x_0)\mathrm{d}\alpha$$

make an identical acreage. Thus

$$\int_{x_0}^{+\infty} \sup_{y \ge x} \mu(y)\mathrm{d}x = \int_0^1 (\sup \mu^{-1}(\alpha) - x_0)\mathrm{d}\alpha = \int_0^1 \sup \mu^{-1}(\alpha)\mathrm{d}\alpha - x_0.$$

Similarly, we may prove

$$\int_{-\infty}^{x_0} \sup_{y \le x} \mu(y)\mathrm{d}x = \int_0^1 (x_0 - \inf \mu^{-1}(\alpha))\mathrm{d}\alpha = x_0 - \int_0^1 \inf \mu^{-1}(\alpha)\mathrm{d}\alpha.$$

It follows from (9.129) that

$$
\begin{aligned}
E[\xi] &= x_0 + \frac{1}{2}\int_{x_0}^{+\infty} \sup_{y\ge x}\mu(y)\mathrm{d}x - \frac{1}{2}\int_{-\infty}^{x_0} \sup_{y\le x}\mu(y)\mathrm{d}x \\
&= x_0 + \frac{1}{2}\left(\int_0^1 \sup\mu^{-1}(\alpha)\mathrm{d}\alpha - x_0\right) - \frac{1}{2}\left(x_0 - \int_0^1 \inf\mu^{-1}(\alpha)\mathrm{d}\alpha\right) \\
&= \frac{1}{2}\int_0^1 (\inf\mu^{-1}(\alpha) + \sup\mu^{-1}(\alpha))\mathrm{d}\alpha.
\end{aligned}
$$

The theorem is thus verified.

Theorem 9.30 *(Liu [133]) Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent uncertain sets with regular membership functions $\mu_1, \mu_2, \cdots, \mu_n$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then the uncertain set*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{9.133}$$

has an expected value

$$E[\xi] = \frac{1}{2}\int_0^1 \left(\mu_l^{-1}(\alpha) + \mu_r^{-1}(\alpha)\right)\mathrm{d}\alpha \tag{9.134}$$

where $\mu_l^{-1}(\alpha)$ and $\mu_r^{-1}(\alpha)$ are determined by

$$\mu_l^{-1}(\alpha) = f(\mu_{1l}^{-1}(\alpha), \cdots, \mu_{ml}^{-1}(\alpha), \mu_{m+1,r}^{-1}(\alpha), \cdots, \mu_{nr}^{-1}(\alpha)), \tag{9.135}$$

$$\mu_r^{-1}(\alpha) = f(\mu_{1r}^{-1}(\alpha), \cdots, \mu_{mr}^{-1}(\alpha), \mu_{m+1,l}^{-1}(\alpha), \cdots, \mu_{nl}^{-1}(\alpha)). \tag{9.136}$$

Proof: It follows from Theorems 9.26 and 9.29 immediately.

Exercise 9.13: Let ξ and η be independent and nonnegative uncertain sets with regular membership functions μ and ν, respectively. Show that

$$E[\xi\eta] = \frac{1}{2}\int_0^1 \left(\mu_l^{-1}(\alpha)\nu_l^{-1}(\alpha) + \mu_r^{-1}(\alpha)\nu_r^{-1}(\alpha)\right)\mathrm{d}\alpha. \tag{9.137}$$

Exercise 9.14: Let ξ and η be independent and positive uncertain sets with regular membership functions μ and ν, respectively. Show that

$$E\left[\frac{\xi}{\eta}\right] = \frac{1}{2}\int_0^1 \left(\frac{\mu_l^{-1}(\alpha)}{\nu_r^{-1}(\alpha)} + \frac{\mu_r^{-1}(\alpha)}{\nu_l^{-1}(\alpha)}\right)\mathrm{d}\alpha. \tag{9.138}$$

Exercise 9.15: Let ξ and η be independent and positive uncertain sets with regular membership functions μ and ν, respectively. Show that

$$E\left[\frac{\xi}{\xi+\eta}\right] = \frac{1}{2}\int_0^1 \left(\frac{\mu_l^{-1}(\alpha)}{\mu_l^{-1}(\alpha)+\nu_r^{-1}(\alpha)} + \frac{\mu_r^{-1}(\alpha)}{\mu_r^{-1}(\alpha)+\nu_l^{-1}(\alpha)}\right)\mathrm{d}\alpha. \tag{9.139}$$

Linearity of Expected Value Operator

Theorem 9.31 *(Liu [133]) Let ξ and η be independent uncertain sets with finite expected values. Then for any real numbers a and b, we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \tag{9.140}$$

Proof: Denote the membership functions of ξ and η by μ and ν, respectively. Then

$$E[\xi] = \frac{1}{2}\int_0^1 \left(\inf \mu^{-1}(\alpha) + \sup \mu^{-1}(\alpha)\right) \mathrm{d}\alpha,$$

$$E[\eta] = \frac{1}{2}\int_0^1 \left(\inf \nu^{-1}(\alpha) + \sup \nu^{-1}(\alpha)\right) \mathrm{d}\alpha.$$

STEP 1: We first prove $E[a\xi] = aE[\xi]$. The product $a\xi$ has an inverse membership function,

$$\lambda^{-1}(\alpha) = a\mu^{-1}(\alpha).$$

It follows from Theorem 9.29 that

$$\begin{aligned} E[a\xi] &= \frac{1}{2}\int_0^1 \left(\inf \lambda^{-1}(\alpha) + \sup \lambda^{-1}(\alpha)\right) \mathrm{d}\alpha \\ &= \frac{a}{2}\int_0^1 \left(\inf \mu^{-1}(\alpha) + \sup \mu^{-1}(\alpha)\right) \mathrm{d}\alpha = aE[\xi]. \end{aligned}$$

STEP 2: We then prove $E[\xi + \eta] = E[\xi] + E[\eta]$. The sum $\xi + \eta$ has an inverse membership function,

$$\lambda^{-1}(\alpha) = \mu^{-1}(\alpha) + \nu^{-1}(\alpha).$$

It follows from Theorem 9.29 that

$$\begin{aligned} E[\xi + \eta] &= \frac{1}{2}\int_0^1 \left(\inf \lambda^{-1}(\alpha) + \sup \lambda^{-1}(\alpha)\right) \mathrm{d}\alpha \\ &= \frac{1}{2}\int_0^1 \left(\inf \mu^{-1}(\alpha) + \sup \mu^{-1}(\alpha)\right) \mathrm{d}\alpha \\ &\quad + \frac{1}{2}\int_0^1 \left(\inf \nu^{-1}(\alpha) + \sup \nu^{-1}(\alpha)\right) \mathrm{d}\alpha \\ &= E[\xi] + E[\eta]. \end{aligned}$$

STEP 3: Finally, for any real numbers a and b, it follows from Steps 1 and 2 that

$$E[a\xi + b\eta] = E[a\xi] + E[b\eta] = aE[\xi] + bE[\eta].$$

The theorem is proved.

9.7 Variance

The variance of uncertain set provides a degree of the spread of the membership function around its expected value.

Definition 9.11 *(Liu [130]) Let ξ be an uncertain set with finite expected value e. Then the variance of ξ is defined by*

$$V[\xi] = E[(\xi - e)^2]. \tag{9.141}$$

This definition says that the variance is just the expected value of $(\xi - e)^2$. Since $(\xi - e)^2$ is a nonnegative uncertain set, we also have

$$V[\xi] = \int_0^{+\infty} \mathcal{M}\{(\xi - e)^2 \succeq x\}\mathrm{d}x. \tag{9.142}$$

Please note that $(\xi - e)^2 \succeq x$ represents "$(\xi - e)^2$ is imaginarily included in $[x, +\infty)$". What is the appropriate value of $\mathcal{M}\{(\xi - e)^2 \succeq x\}$? Intuitively, it is too conservative if we take the value $\mathcal{M}\{(\xi - e)^2 \geq x\}$, and it is too adventurous if we take the value $1 - \mathcal{M}\{(\xi - e)^2 < x\}$. Thus we assign $\mathcal{M}\{(\xi - e)^2 \succeq x\}$ the middle value between them. That is,

$$\mathcal{M}\{(\xi - e)^2 \succeq x\} = \frac{1}{2}\left(\mathcal{M}\{(\xi - e)^2 \geq x\} + 1 - \mathcal{M}\{(\xi - e)^2 < x\}\right). \tag{9.143}$$

Theorem 9.32 *If ξ is an uncertain set with finite expected value, a and b are real numbers, then*

$$V[a\xi + b] = a^2 V[\xi]. \tag{9.144}$$

Proof: If ξ has an expected value e, then $a\xi + b$ has an expected value $ae + b$. It follows from the definition of variance that

$$V[a\xi + b] = E\left[(a\xi + b - ae - b)^2\right] = a^2 E[(\xi - e)^2] = a^2 V[\xi].$$

Theorem 9.33 *Let ξ be an uncertain set with expected value e. Then $V[\xi] = 0$ if and only if $\xi = \{e\}$ almost surely.*

Proof: We first assume $V[\xi] = 0$. It follows from the equation (9.142) that

$$\int_0^{+\infty} \mathcal{M}\{(\xi - e)^2 \succeq x\}\mathrm{d}x = 0$$

which implies $\mathcal{M}\{(\xi - e)^2 \succeq x\} = 0$ for any $x > 0$. Hence $\mathcal{M}\{\xi = \{e\}\} = 1$. Conversely, assume $\mathcal{M}\{\xi = \{e\}\} = 1$. Then we have $\mathcal{M}\{(\xi - e)^2 \succeq x\} = 0$ for any $x > 0$. Thus

$$V[\xi] = \int_0^{+\infty} \mathcal{M}\{(\xi - e)^2 \succeq x\}\mathrm{d}x = 0.$$

The theorem is proved.

How to Obtain Variance from Membership Function?

Let ξ be an uncertain set with membership function μ. In order to calculate its variance by (9.142), we must determine the value of $\mathcal{M}\{(\xi-e)^2 \succeq x\}$ from the membership function μ.

Theorem 9.34 *Let ξ be an uncertain set with membership function μ. Then for any real numbers e and x, we have*

$$\mathcal{M}\{(\xi-e)^2 \succeq x\} = \frac{1}{2}\left(\sup_{(y-e)^2 \ge x} \mu(y) + 1 - \sup_{(y-e)^2 < x} \mu(y)\right). \tag{9.145}$$

Proof: Since ξ is an uncertain set with membership function μ, it follows from the measure inversion formula that for any real numbers e and x, we have

$$\mathcal{M}\{(\xi-e)^2 \ge x\} = 1 - \sup_{(y-e)^2 < x} \mu(y),$$

$$\mathcal{M}\{(\xi-e)^2 < x\} = 1 - \sup_{(y-e)^2 \ge x} \mu(y).$$

The equation (9.145) is thus proved by (9.143).

Theorem 9.35 *Let ξ be an uncertain set with membership function μ and finite expected value e. Then*

$$V[\xi] = \frac{1}{2}\int_0^{+\infty}\left(\sup_{(y-e)^2 \ge x} \mu(y) + 1 - \sup_{(y-e)^2 < x} \mu(y)\right)\mathrm{d}x. \tag{9.146}$$

Proof: This theorem follows from (9.142) and (9.145) immediately.

9.8 Entropy

This section provides a definition of entropy to characterize the uncertainty of uncertain sets.

Definition 9.12 *(Liu [130]) Suppose that ξ is an uncertain set with membership function μ. Then its entropy is defined by*

$$H[\xi] = \int_{-\infty}^{+\infty} S(\mu(x))\mathrm{d}x \tag{9.147}$$

where $S(t) = -t\ln t - (1-t)\ln(1-t)$.

Remark 9.9: Note that the entropy (9.147) has the same form with de Luca and Termini's entropy for fuzzy set [32].

Remark 9.10: If ξ is a discrete uncertain set taking values in $\{x_1, x_2, \cdots\}$, then the entropy becomes

$$H[\xi] = \sum_{i=1}^{\infty} S(\mu(x_i)). \tag{9.148}$$

Example 9.12: A crisp set A of real numbers is a special uncertain set $\xi(\gamma) \equiv A$. Then its membership function is

$$\mu(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

and entropy is

$$H[\xi] = \int_{-\infty}^{+\infty} S(\mu(x))\mathrm{d}x = \int_{-\infty}^{+\infty} 0\mathrm{d}x = 0.$$

Exercise 9.16: Let $\xi = (a, b, c)$ be a triangular uncertain set. Show that its entropy is

$$H[\xi] = \frac{c-a}{2}. \tag{9.149}$$

Exercise 9.17: Let $\xi = (a, b, c, d)$ be a trapezoidal uncertain set. Show that its entropy is

$$H[\xi] = \frac{b-a+d-c}{2}. \tag{9.150}$$

Theorem 9.36 *Let ξ be an uncertain set. Then $H[\xi] \geq 0$ and equality holds if ξ is essentially a crisp set.*

Proof: The nonnegativity is clear. In addition, when an uncertain set tends to a crisp set, its entropy tends to the minimum value 0.

Theorem 9.37 *Let ξ be an uncertain set on the interval $[a, b]$. Then*

$$H[\xi] \leq (b-a)\ln 2 \tag{9.151}$$

and equality holds if ξ has a membership function $\mu(x) = 0.5$ on $[a, b]$.

Proof: The theorem follows from the fact that the function $S(t)$ reaches its maximum value $\ln 2$ at $t = 0.5$.

Theorem 9.38 *Let ξ be an uncertain set, and let ξ^c be its complement. Then*

$$H[\xi^c] = H[\xi]. \tag{9.152}$$

Proof: Write the membership function of ξ by μ. Then its complement ξ^c has a membership function $1-\mu(x)$. It follows from the definition of entropy that

$$H[\xi^c]=\int_{-\infty}^{+\infty} S\left(1-\mu(x)\right)\mathrm{d}x=\int_{-\infty}^{+\infty} S(\mu(x))\mathrm{d}x=H[\xi].$$

The theorem is proved.

Theorem 9.39 *(Yao [249]) Let ξ be an uncertain set with regular membership function μ. Then*

$$H[\xi]=\int_0^1 (\mu_l^{-1}(\alpha)-\mu_r^{-1}(\alpha))\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha. \tag{9.153}$$

Proof: It is clear that $S(\alpha)=-\alpha\ln\alpha-(1-\alpha)\ln(1-\alpha)$ is a derivable function whose derivative is

$$S'(\alpha)=-\ln\frac{\alpha}{1-\alpha}.$$

Let x_0 be a point such that $\mu(x_0)=1$. Then we have

$$\begin{aligned}H[\xi]&=\int_{-\infty}^{+\infty} S(\mu(x))\mathrm{d}x=\int_{-\infty}^{x_0} S(\mu(x))\mathrm{d}x+\int_{x_0}^{+\infty} S(\mu(x))\mathrm{d}x\\&=\int_{-\infty}^{x_0}\int_0^{\mu(x)} S'(\alpha)\mathrm{d}\alpha\mathrm{d}x+\int_{x_0}^{+\infty}\int_0^{\mu(x)} S'(\alpha)\mathrm{d}\alpha\mathrm{d}x.\end{aligned}$$

It follows from Fubini theorem that

$$\begin{aligned}H[\xi]&=\int_0^1\int_{\mu_l^{-1}(\alpha)}^{x_0} S'(\alpha)\mathrm{d}x\mathrm{d}\alpha+\int_0^1\int_{x_0}^{\mu_r^{-1}(\alpha)} S'(\alpha)\mathrm{d}x\mathrm{d}\alpha\\&=\int_0^1 (x_0-\mu_l^{-1}(\alpha))S'(\alpha)\mathrm{d}\alpha+\int_0^1 (\mu_r^{-1}(\alpha)-x_0)S'(\alpha)\mathrm{d}\alpha\\&=\int_0^1 (\mu_r^{-1}(\alpha)-\mu_l^{-1}(\alpha))S'(\alpha)\mathrm{d}\alpha\\&=\int_0^1 (\mu_l^{-1}(\alpha)-\mu_r^{-1}(\alpha))\ln\frac{\alpha}{1-\alpha}\mathrm{d}\alpha.\end{aligned}$$

The theorem is verified.

Positive Linearity of Entropy

Theorem 9.40 *(Yao [249]) Let ξ and η be independent uncertain sets. Then for any real numbers a and b, we have*

$$H[a\xi+b\eta]=|a|H[\xi]+|b|H[\eta]. \tag{9.154}$$

Proof: Without loss of generality, assume the uncertain sets ξ and η have regular membership functions μ and ν, respectively.

Step 1: We prove $H[a\xi] = |a|H[\xi]$. If $a > 0$, then the left and right inverse membership functions of $a\xi$ are

$$\lambda_l^{-1}(\alpha) = a\mu_l^{-1}(\alpha), \quad \lambda_r^{-1}(\alpha) = a\mu_r^{-1}(\alpha).$$

It follows from Theorem 9.39 that

$$H[a\xi] = \int_0^1 (a\mu_l^{-1}(\alpha) - a\mu_r^{-1}(\alpha)) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha = aH[\xi] = |a|H[\xi].$$

If $a = 0$, then we immediately have $H[a\xi] = 0 = |a|H[\xi]$. If $a < 0$, then we have

$$\lambda_l^{-1}(\alpha) = a\mu_r^{-1}(\alpha), \quad \lambda_r^{-1}(\alpha) = a\mu_l^{-1}(\alpha)$$

and

$$H[a\xi] = \int_0^1 (a\mu_r^{-1}(\alpha) - a\mu_l^{-1}(\alpha)) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha = (-a)H[\xi] = |a|H[\xi].$$

Thus we always have $H[a\xi] = |a|H[\xi]$.

Step 2: We prove $H[\xi + \eta] = H[\xi] + H[\eta]$. Note that the left and right inverse membership functions of $\xi + \eta$ are

$$\lambda_l^{-1}(\alpha) = \mu_l^{-1}(\alpha) + \nu_l^{-1}(\alpha), \quad \lambda_r^{-1}(\alpha) = \mu_r^{-1}(\alpha) + \nu_r^{-1}(\alpha).$$

It follows from Theorem 9.39 that

$$\begin{aligned} H[\xi + \eta] &= \int_0^1 (\lambda_l^{-1}(\alpha) - \lambda_r^{-1}(\alpha)) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha \\ &= \int_0^1 (\mu_l^{-1}(\alpha) + \nu_l^{-1}(\alpha) - \mu_r^{-1}(\alpha) - \nu_r^{-1}(\alpha)) \ln \frac{\alpha}{1-\alpha} \mathrm{d}\alpha \\ &= H[\xi] + H[\eta]. \end{aligned}$$

Step 3: Finally, for any real numbers a and b, it follows from Steps 1 and 2 that

$$H[a\xi + b\eta] = H[a\xi] + H[b\eta] = |a|H[\xi] + |b|H[\eta].$$

The theorem is proved.

Exercise 9.18: Let ξ be an uncertain set, and let A be a crisp set. Show that

$$H[\xi + A] = H[\xi]. \tag{9.155}$$

That is, the entropy is invariant under arbitrary translations.

9.9 Distance

Definition 9.13 *(Liu [130]) The distance between uncertain sets* ξ *and* η *is defined as*

$$d(\xi,\eta) = E[|\xi-\eta|]. \tag{9.156}$$

That is, the distance between ξ and η is just the expected value of $|\xi-\eta|$. Since $|\xi-\eta|$ is a nonnegative uncertain set, we have

$$d(\xi,\eta) = \int_0^{+\infty} \mathcal{M}\{|\xi-\eta| \succeq x\}\mathrm{d}x. \tag{9.157}$$

Please note that $|\xi-\eta| \succeq x$ represents "$|\xi-\eta|$ is imaginarily included in $[x,+\infty)$". What is the appropriate value of $\mathcal{M}\{|\xi-\eta| \succeq x\}$? Intuitively, it is too conservative if we take the value $\mathcal{M}\{|\xi-\eta| \ge x\}$, and it is too adventurous if we take the value $1-\mathcal{M}\{|\xi-\eta| < x\}$. Thus we assign $\mathcal{M}\{|\xi-\eta| \succeq x\}$ the middle value between them. That is,

$$\mathcal{M}\{|\xi-\eta| \succeq x\} = \frac{1}{2}\left(\mathcal{M}\{|\xi-\eta| \ge x\} + 1 - \mathcal{M}\{|\xi-\eta| < x\}\right). \tag{9.158}$$

Theorem 9.41 *Let* ξ *and* η *be uncertain sets. Then for any real number* x*, we have*

$$\mathcal{M}\{|\xi-\eta| \succeq x\} = \frac{1}{2}\left(\sup_{|y|\ge x}\lambda(y) + 1 - \sup_{|y|<x}\lambda(y)\right) \tag{9.159}$$

where λ *is the membership function of* $\xi-\eta$*.*

Proof: Since $\xi-\eta$ is an uncertain set with membership function λ, it follows from the measure inversion formula that for any real number x, we have

$$\mathcal{M}\{|\xi-\eta| \ge x\} = 1 - \sup_{|y|<x}\mu(y),$$

$$\mathcal{M}\{|\xi-\eta| < x\} = 1 - \sup_{|y|\ge x}\mu(y).$$

The equation (9.159) is thus proved by (9.158).

Theorem 9.42 *Let* ξ *and* η *be uncertain sets. Then the distance between* ξ *and* η *is*

$$d(\xi,\eta) = \frac{1}{2}\int_0^{+\infty}\left(\sup_{|y|\ge x}\lambda(y) + 1 - \sup_{|y|<x}\lambda(y)\right)\mathrm{d}x \tag{9.160}$$

where λ *is the membership function of* $\xi-\eta$*.*

Proof: The theorem follows from (9.157) and (9.159) immediately.

9.10 Conditional Membership Function

What is the conditional membership function of an uncertain set ξ after it has been learned that some event A has occurred? This section will answer this question. At first, it follows from the definition of conditional uncertain measure that

$$\mathcal{M}\{B\subset\xi|A\}=\begin{cases}\dfrac{\mathcal{M}\{(B\subset\xi)\cap(\xi\subset A)\}}{\mathcal{M}\{\xi\subset A\}}, & \text{if } \dfrac{\mathcal{M}\{(B\subset\xi)\cap(\xi\subset A)\}}{\mathcal{M}\{\xi\subset A\}}<0.5\\ 1-\dfrac{\mathcal{M}\{(B\not\subset\xi)\cap(\xi\subset A)\}}{\mathcal{M}\{\xi\subset A\}}, & \text{if } \dfrac{\mathcal{M}\{(B\not\subset\xi)\cap(\xi\subset A)\}}{\mathcal{M}\{\xi\subset A\}}<0.5\\ 0.5, & \text{otherwise,}\end{cases}$$

$$\mathcal{M}\{\xi\subset B|A\}=\begin{cases}\dfrac{\mathcal{M}\{(\xi\subset B)\cap(\xi\subset A)\}}{\mathcal{M}\{\xi\subset A\}}, & \text{if } \dfrac{\mathcal{M}\{(\xi\subset B)\cap(\xi\subset A)\}}{\mathcal{M}\{\xi\subset A\}}<0.5\\ 1-\dfrac{\mathcal{M}\{(\xi\not\subset B)\cap(\xi\subset A)\}}{\mathcal{M}\{\xi\subset A\}}, & \text{if } \dfrac{\mathcal{M}\{(\xi\not\subset B)\cap(\xi\subset A)\}}{\mathcal{M}\{\xi\subset A\}}<0.5\\ 0.5, & \text{otherwise.}\end{cases}$$

Definition 9.14 *Let ξ be an uncertain set, and let A be an event with $\mathcal{M}\{A\}>0$. Then the conditional uncertain set ξ given A is said to have a membership function $\mu(x|A)$ if for any Borel set B, we have*

$$\mathcal{M}\{B\subset\xi|A\}=\inf_{x\in B}\mu(x|A), \tag{9.161}$$

$$\mathcal{M}\{\xi\subset B|A\}=1-\sup_{x\in B^c}\mu(x|A). \tag{9.162}$$

9.11 Uncertain Statistics

In order to determine the membership function of uncertain set, Liu [130] designed a questionnaire survey for collecting expert's experimental data, and introduced the empirical membership function (i.e., linear interpolation method) and the principle of least squares.

Expert's Experimental Data

Expert's experimental data were suggested by Liu [130] to represent expert's knowledge about the membership function to be determined. The first step is to ask the domain expert to choose a possible point x that the uncertain set ξ may contain, and then quiz him

$$\text{"How likely does } x \text{ belong to } \xi\text{?"} \tag{9.163}$$

Assume the expert's belief degree is α in uncertain measure. Note that the expert's belief degree of x not belonging to ξ must be $1-\alpha$ due to the duality of uncertain measure. An expert's experimental data (x,α) is thus acquired from the domain expert. Repeating the above process, the following expert's experimental data are obtained by the questionnaire,

$$(x_1,\alpha_1),\ (x_2,\alpha_2),\ \cdots,\ (x_n,\alpha_n). \tag{9.164}$$

Empirical Membership Function

How do we determine the membership function for an uncertain set? The first method is the linear interpolation method developed by Liu [130]. Assume that we have obtained a set of expert's experimental data

$$(x_1,\alpha_1), (x_2,\alpha_2), \cdots, (x_n,\alpha_n). \tag{9.165}$$

Without loss of generality, we also assume $x_1 < x_2 < \cdots < x_n$. Based on those expert's experimental data, an empirical membership function is determined as follows,

$$\mu(x) = \begin{cases} \alpha_i + \dfrac{(\alpha_{i+1}-\alpha_i)(x-x_i)}{x_{i+1}-x_i}, & \text{if } x_i \le x \le x_{i+1},\ 1 \le i < n \\ 0, & \text{otherwise.} \end{cases}$$

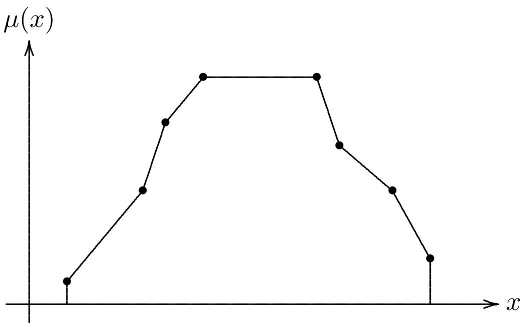


Figure 9.14: Empirical Membership Function $\mu(x)$

Principle of Least Squares

Principle of least squares was first employed to determine membership function by Liu [130]. Assume that a membership function to be determined has a known functional form $\mu(x|\theta)$ with an unknown parameter θ. In order to estimate the parameter θ, we may employ the principle of least squares that minimizes the sum of the squares of the distance of the expert's experimental data to the membership function. If the expert's experimental data

$$(x_1, \alpha_1),\ (x_2, \alpha_2),\ \cdots,\ (x_n, \alpha_n) \tag{9.166}$$

are obtained, then we have

$$\min_{\theta} \sum_{i=1}^{n} (\mu(x_i|\theta) - \alpha_i)^2. \tag{9.167}$$

The optimal solution $\widehat{\theta}$ of (9.167) is called the least squares estimate of θ, and then the least squares membership function is $\mu(x|\widehat{\theta})$.

Example 9.13: Assume that a membership function has a trapezoidal form (a, b, c, d). We also assume the following expert's experimental data,

$$(1, 0.15),\ (2, 0.45),\ (3, 0.90),\ (6, 0.85),\ (7, 0.60),\ (8, 0.20). \tag{9.168}$$

The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may yield that the least squares membership function has a trapezoidal form $(0.6667, 3.3333, 5.6154, 8.6923)$.

What is "about 100km"?

Let us pay attention to the concept of "about 100km". When we are interested in what distances can be considered "about 100km", it is reasonable to regard such a concept as an uncertain set. In order to determine the membership function of "about 100km", a questionnaire survey was made for collecting expert's experimental data. The consultation process is as follows:

Q1: May I ask you what distances belong to "about 100km"? What do you think is the minimum distance?

A1: 80km. *(an expert's experimental data* $(80, 0)$ *is acquired)*

Q2: What do you think is the maximum distance?

A2: 120km. *(an expert's experimental data* $(120, 0)$ *is acquired)*

Q3: What distance do you think belongs to "about 100km"?

A3: 95km.

Q4: What is the belief degree that 95km belongs to "about 100km"?

A4: 1. *(an expert's experimental data* $(95, 1)$ *is acquired)*

Q5: Is there another distance that belongs to "about 100km"?

A5: 105km.

Q6: What is the belief degree that 105km belongs to “about 100km”?

A6: 1. *(an expert’s experimental data* $(105, 1)$ *is acquired)*

Q7: Is there another distance that belongs to “about 100km”?

A7: 90km.

Q8: What is the belief degree that 90km belongs to “about 100km”?

A8: 0.5. *(an expert’s experimental data* $(90, 0.5)$ *is acquired)*

Q9: Is there another distance that belongs to “about 100km”?

A9: 110km.

Q10: What is the belief degree that 110km belongs to “about 100km”?

A10: 0.5. *(an expert’s experimental data* $(110, 0.5)$ *is acquired)*

Q11: Is there another distance that belongs to “about 100km”?

A11: No idea.

Until now six expert’s experimental data $(80, 0)$, $(90, 0.5)$, $(95, 1)$, $(105, 1)$, $(110, 0.5)$, $(120, 0)$ are acquired from the domain expert. Based on those expert’s experimental data, an empirical membership function of “about 100km” is produced and shown by Figure 9.15.

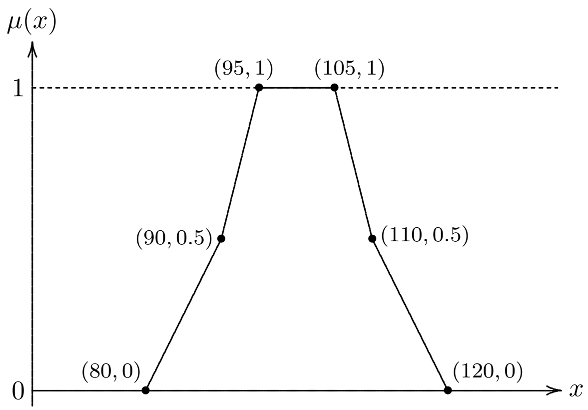


Figure 9.15: Empirical Membership Function of “about 100km”

9.12 Bibliographic Notes

In order to model unsharp concepts like “young”, “tall” and “most”, the concept of uncertain set was proposed by Liu [127] in 2010, and the concepts of membership function and inverse membership function were presented by Liu [133] in 2012. Following that, Liu [136] defined the independence of uncertain sets, and provided an operational law through membership functions in 2013.

The expected value of uncertain set was defined by Liu [127]. Then Liu [129] gave a formula for caluculating the expected value by membership function, and Liu [133] provided a formula by inverse membership function. Based on expected value operator, Liu [130] presented the concept of variance and distance between uncertain sets.

The first concept of entropy was presented by Liu [130] for measuring the uncertainty of uncertain set. As extensions of entropy, Wang and Ha [229] suggested a quadratic entropy, and Yao [249] proposed a cross entropy for comparing a membership function against a reference membership function.

In order to determine membership functions, a questionnaire survey for collecting expert’s experimental data was designed by Liu [130]. Based on expert’s experimental data, Liu [130] also suggested linear interpolation method and principle of least squares to determine membership functions. When multiple domain experts are available, Delphi method was introduced to uncertain statistics by Wang and Wang [231].

Chapter 10

Uncertain Logic

Uncertain logic is a methodology for calculating the truth values of uncertain propositions via uncertain set theory. This chapter will introduce individual feature data, uncertain quantifier, uncertain subject, uncertain predicate, uncertain proposition, and truth value. Uncertain logic may provide a flexible means for extracting linguistic summary from a collection of raw data.

10.1 Individual Feature Data

At first, we should have a universe A of individuals we are talking about. Without loss of generality, we may assume that A consists of n individuals and is represented by

$$A = \{a_1, a_2, \cdots, a_n\}. \tag{10.1}$$

In order to deal with the universe A, we should have feature data of all individuals $a_1, a_2, \cdots, a_n$. When we talk about "those days are warm", we should know the individual feature data of all days, for example,

$$A = \{22, 23, 25, 28, 30, 32, 36\} \tag{10.2}$$

whose elements are temperatures in centigrades. When we talk about "those students are young", we should know the individual feature data of all students, for example,

$$A = \{21, 22, 22, 23, 24, 25, 26, 27, 28, 30, 32, 35, 36, 38, 40\} \tag{10.3}$$

whose elements are ages in years. When we talk about "those sportsmen are tall", we should know the individual feature data of all sportsmen, for example,

$$A = \left\{\begin{array}{l} 175, 178, 178, 180, 183, 184, 186, 186 \\ 188, 190, 192, 192, 193, 194, 195, 196 \end{array}\right\} \tag{10.4}$$

whose elements are heights in centimeters.

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5_11

Sometimes the individual feature data are represented by vectors rather a scalar number. When we talk about "those young students are tall", we should know the individual feature data of all students, for example,

$$A = \left\{ \begin{array}{l} (24,185),\ (25,190),\ (26,184),\ (26,170),\ (27,187),\ (27,188) \\ (28,160),\ (30,190),\ (32,185),\ (33,176),\ (35,185),\ (36,188) \\ (38,164),\ (38,178),\ (39,182),\ (40,186),\ (42,165),\ (44,170) \end{array} \right\} \tag{10.5}$$

whose elements are ages and heights in years and centimeters, respectively.

10.2 Uncertain Quantifier

If we want to represent all individuals in the universe A, we use the universal quantifier ($\forall$) and

$$\forall = \text{“for all”}. \tag{10.6}$$

If we want to represent some (at least one) individuals, we use the existential quantifier ($\exists$) and

$$\exists = \text{“there exists at least one”}. \tag{10.7}$$

In addition to the two quantifiers, there are numerous imprecise quantifiers in human language, for example, *almost all*, *almost none*, *many*, *several*, *some*, *most*, *a few*, *about half*. This section will model them by the concept of uncertain quantifier.

Definition 10.1 *(Liu [130]) Uncertain quantifier is an uncertain set representing the number of individuals.*

Example 10.1: The universal quantifier ($\forall$) on the universe A is a special uncertain quantifier,

$$\forall \equiv \{n\} \tag{10.8}$$

whose membership function is

$$\lambda(x) = \begin{cases} 1, & \text{if } x = n \\ 0, & \text{otherwise.} \end{cases} \tag{10.9}$$

Example 10.2: The existential quantifier ($\exists$) on the universe A is a special uncertain quantifier,

$$\exists \equiv \{1, 2, \cdots, n\} \tag{10.10}$$

whose membership function is

$$\lambda(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{otherwise.} \end{cases} \tag{10.11}$$

Example 10.3: The quantifier "*there does not exist one*" on the universe A is a special uncertain quantifier

$$\mathcal{Q} \equiv \{0\} \tag{10.12}$$

whose membership function is

$$\lambda(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{10.13}$$

Example 10.4: The quantifier "*there exist exactly m*" on the universe A is a special uncertain quantifier

$$\mathcal{Q} \equiv \{m\} \tag{10.14}$$

whose membership function is

$$\lambda(x) = \begin{cases} 1, & \text{if } x = m \\ 0, & \text{otherwise.} \end{cases} \tag{10.15}$$

Example 10.5: The quantifier "*there exist at least m*" on the universe A is a special uncertain quantifier

$$\mathcal{Q} \equiv \{m, m+1, \cdots, n\} \tag{10.16}$$

whose membership function is

$$\lambda(x) = \begin{cases} 1, & \text{if } m \le x \le n \\ 0, & \text{if } 0 \le x < m. \end{cases} \tag{10.17}$$

Example 10.6: The quantifier "*there exist at most m*" on the universe A is a special uncertain quantifier

$$\mathcal{Q} \equiv \{0, 1, 2, \cdots, m\} \tag{10.18}$$

whose membership function is

$$\lambda(x) = \begin{cases} 1, & \text{if } 0 \le x \le m \\ 0, & \text{if } m < x \le n. \end{cases} \tag{10.19}$$

Example 10.7: The uncertain quantifier $\mathcal{Q}$ of "*almost all*" on the universe A may have a membership function

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \le x \le n-5 \\ (x-n+5)/3, & \text{if } n-5 \le x \le n-2 \\ 1, & \text{if } n-2 \le x \le n. \end{cases} \tag{10.20}$$

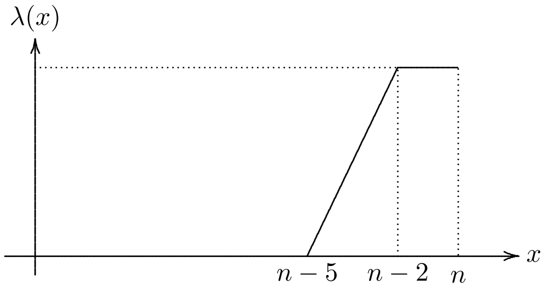


Figure 10.1: Membership Function of Quantifier "*almost all*"

Example 10.8: The uncertain quantifier $\mathcal{Q}$ of "*almost none*" on the universe A may have a membership function

$$\lambda(x) = \begin{cases} 1, & \text{if } 0 \le x \le 2 \\ (5-x)/3, & \text{if } 2 \le x \le 5 \\ 0, & \text{if } 5 \le x \le n. \end{cases} \tag{10.21}$$

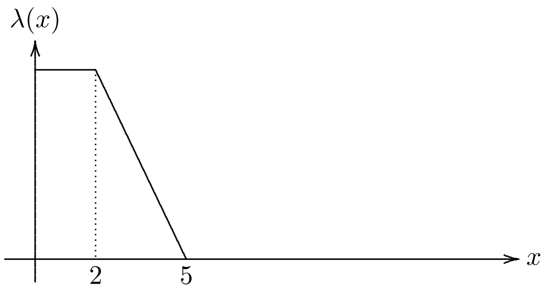


Figure 10.2: Membership Function of Quantifier "*almost none*"

Example 10.9: The uncertain quantifier $\mathcal{Q}$ of "*about 10*" on the universe A may have a membership function

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \le x \le 7 \\ (x-7)/2, & \text{if } 7 \le x \le 9 \\ 1, & \text{if } 9 \le x \le 11 \\ (13-x)/2, & \text{if } 11 \le x \le 13 \\ 0, & \text{if } 13 \le x \le n. \end{cases} \tag{10.22}$$

Example 10.10: In many cases, it is more convenient for us to use a percentage than an absolute quantity. For example, we may use the uncertain

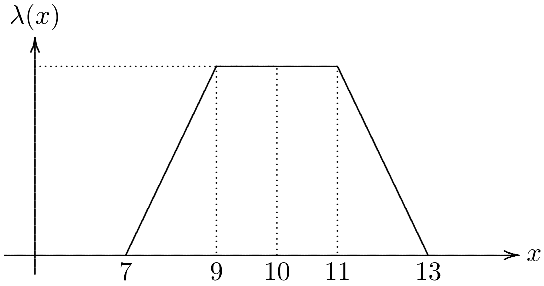


Figure 10.3: Membership Function of Quantifier "*about 10*"

quantifier $\mathbb{Q}$ of "*about 70%*". In this case, a possible membership function of $\mathbb{Q}$ is

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \le x \le 0.6 \\ 20(x-0.6), & \text{if } 0.6 \le x \le 0.65 \\ 1, & \text{if } 0.65 \le x \le 0.75 \\ 20(0.8-x), & \text{if } 0.75 \le x \le 0.8 \\ 0, & \text{if } 0.8 \le x \le 1. \end{cases} \quad (10.23)$$

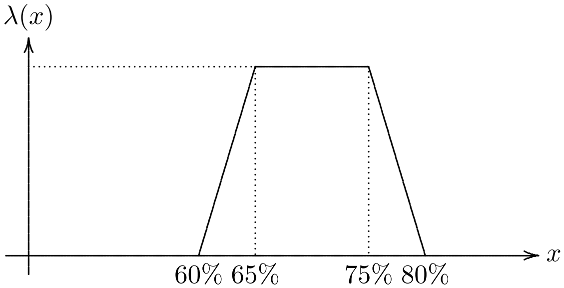


Figure 10.4: Membership Function of Quantifier "*about 70%*"

Definition 10.2 *An uncertain quantifier is said to be unimodal if its membership function is unimodal.*

Example 10.11: The uncertain quantifiers "almost all", "almost none", "about 10" and "about 70%" are unimodal.

Definition 10.3 *An uncertain quantifier is said to be monotone if its membership function is monotone. Especially, an uncertain quantifier is said to be increasing if its membership function is increasing; and an uncertain quantifier is said to be decreasing if its membership function is decreasing.*

The uncertain quantifiers "almost all" and "almost none" are monotone, but "about 10" and "about 70%" are not monotone. Note that both increasing uncertain quantifiers and decreasing uncertain quantifiers are monotone. In addition, any monotone uncertain quantifiers are unimodal.

Negated Quantifier

What is the negation of an uncertain quantifier? The following definition gives a formal answer.

Definition 10.4 *Let $\mathcal{Q}$ be an uncertain quantifier. Then the negated quantifier $\neg\mathcal{Q}$ is the complement of $\mathcal{Q}$ in the sense of uncertain set, i.e.,*

$$\neg\mathcal{Q} = \mathcal{Q}^c. \tag{10.24}$$

Example 10.12: Let $\forall = \{n\}$ be the universal quantifier. Then its negated quantifier

$$\neg\forall \equiv \{0, 1, 2, \cdots, n-1\}. \tag{10.25}$$

Example 10.13: Let $\exists = \{1, 2, \cdots, n\}$ be the existential quantifier. Then its negated quantifier is

$$\neg\exists \equiv \{0\}. \tag{10.26}$$

Theorem 10.1 *Let $\mathcal{Q}$ be an uncertain quantifier whose membership function is λ. Then the negated quantifier $\neg\mathcal{Q}$ has a membership function*

$$\neg\lambda(x) = 1 - \lambda(x). \tag{10.27}$$

Proof: This theorem follows from the operational law of uncertain set immediately.

Example 10.14: Let $\mathcal{Q}$ be the uncertain quantifier "*almost all*" defined by (10.20). Then its negated quantifier $\neg\mathcal{Q}$ has a membership function

$$\neg\lambda(x) = \begin{cases} 1, & \text{if } 0 \le x \le n-5 \\ (n-x-2)/3, & \text{if } n-5 \le x \le n-2 \\ 0, & \text{if } n-2 \le x \le n. \end{cases} \tag{10.28}$$

Example 10.15: Let $\mathcal{Q}$ be the uncertain quantifier "*about 70%*" defined by (10.23). Then its negated quantifier $\neg\mathcal{Q}$ has a membership function

$$\neg\lambda(x) = \begin{cases} 1, & \text{if } 0 \le x \le 0.6 \\ 20(0.65-x), & \text{if } 0.6 \le x \le 0.65 \\ 0, & \text{if } 0.65 \le x \le 0.75 \\ 20(x-0.75), & \text{if } 0.75 \le x \le 0.8 \\ 1, & \text{if } 0.8 \le x \le 1. \end{cases} \tag{10.29}$$

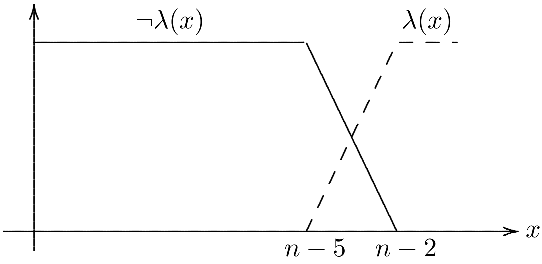


Figure 10.5: Membership Function of Negated Quantifier of "*almost all*"

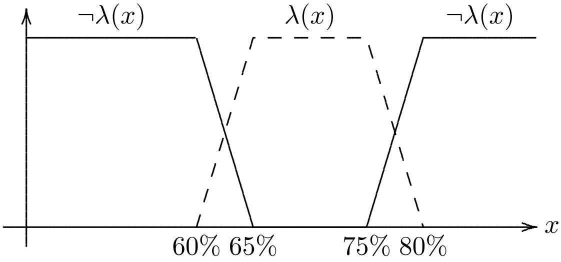


Figure 10.6: Membership Function of Negated Quantifier of "*about 70%*"

Theorem 10.2 *Let $\mathcal{Q}$ be an uncertain quantifier. Then we have $\neg\neg\mathcal{Q} = \mathcal{Q}$.*

Proof: This theorem follows from $\neg\neg\mathcal{Q} = \neg\mathcal{Q}^c = (\mathcal{Q}^c)^c = \mathcal{Q}$.

Theorem 10.3 *If $\mathcal{Q}$ is a monotone uncertain quantifier, then $\neg\mathcal{Q}$ is also monotone. Especially, if $\mathcal{Q}$ is increasing, then $\neg\mathcal{Q}$ is decreasing; if $\mathcal{Q}$ is decreasing, then $\neg\mathcal{Q}$ is increasing.*

Proof: Assume λ is the membership function of $\mathcal{Q}$. Then $\neg\mathcal{Q}$ has a membership function $1 - \lambda(x)$. The theorem follows from this fact immediately.

Dual Quantifier

Definition 10.5 *Let $\mathcal{Q}$ be an uncertain quantifier. Then the dual quantifier of $\mathcal{Q}$ is*

$$\mathcal{Q}^* = \forall - \mathcal{Q}. \tag{10.30}$$

Remark 10.1: Note that $\mathcal{Q}$ and $\mathcal{Q}^*$ are dependent uncertain sets such that $\mathcal{Q} + \mathcal{Q}^* \equiv \forall$. Since the cardinality of the universe A is n, we also have

$$\mathcal{Q}^* = n - \mathcal{Q}. \tag{10.31}$$

Example 10.16: Since $\forall \equiv \{n\}$, we immediately have $\forall^* = \{0\} = \neg\exists$. That is

$$\forall^* \equiv \neg\exists. \tag{10.32}$$

Example 10.17: Since $\neg\forall = \{0, 1, 2, \cdots, n-1\}$, we immediately have $(\neg\forall)^* = \{1, 2, \cdots, n\} = \exists$. That is,

$$(\neg\forall)^* \equiv \exists. \tag{10.33}$$

Example 10.18: Since $\exists \equiv \{1, 2, \cdots, n\}$, we have $\exists^* = \{0, 1, 2, \cdots, n-1\} = \neg\forall$. That is,

$$\exists^* \equiv \neg\forall. \tag{10.34}$$

Example 10.19: Since $\neg\exists = \{0\}$, we immediately have $(\neg\exists)^* = \{n\} = \forall$. That is,

$$(\neg\exists)^* = \forall. \tag{10.35}$$

Theorem 10.4 *Let $\mathcal{Q}$ be an uncertain quantifier whose membership function is λ. Then the dual quantifier $\mathcal{Q}^*$ has a membership function*

$$\lambda^*(x) = \lambda(n - x) \tag{10.36}$$

where n is the cardinality of the universe A.

Proof: This theorem follows from the operational law of uncertain set immediately.

Example 10.20: Let $\mathcal{Q}$ be the uncertain quantifier "*almost all*" defined by (10.20). Then its dual quantifier $\mathcal{Q}^*$ has a membership function

$$\lambda^*(x) = \begin{cases} 1, & \text{if } 0 \le x \le 2 \\ (5 - x)/3, & \text{if } 2 \le x \le 5 \\ 0, & \text{if } 5 \le x \le n. \end{cases} \tag{10.37}$$

Example 10.21: Let $\mathcal{Q}$ be the uncertain quantifier "*about 70%*" defined by (10.23). Then its dual quantifier $\mathcal{Q}^*$ has a membership function

$$\lambda^*(x) = \begin{cases} 0, & \text{if } 0 \le x \le 0.2 \\ 20(x - 0.2), & \text{if } 0.2 \le x \le 0.25 \\ 1, & \text{if } 0.25 \le x \le 0.35 \\ 20(0.4 - x), & \text{if } 0.35 \le x \le 0.4 \\ 0, & \text{if } 0.4 \le x \le 1. \end{cases} \tag{10.38}$$

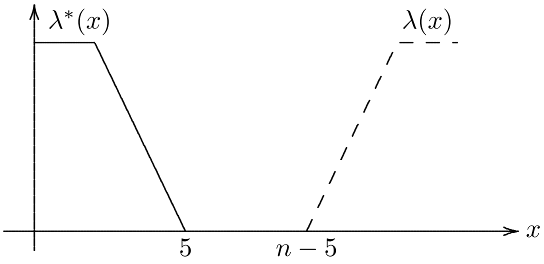


Figure 10.7: Membership Function of Dual Quantifier of "*almost all*"

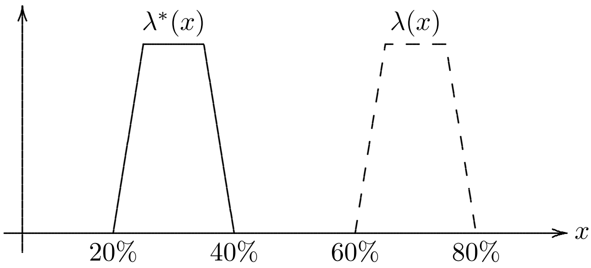


Figure 10.8: Membership Function of Dual Quantifier of "*about 70%*"

Theorem 10.5 *Let* $\mathfrak{Q}$ *be an uncertain quantifier. Then we have* $\mathfrak{Q}^{**} = \mathfrak{Q}$.

Proof: The theorem follows from $\mathfrak{Q}^{**} = \forall - \mathfrak{Q}^* = \forall - (\forall - \mathfrak{Q}) = \mathfrak{Q}$.

Theorem 10.6 *If* $\mathfrak{Q}$ *is a unimodal uncertain quantifier, then* $\mathfrak{Q}^*$ *is also unimodal. Especially, if* $\mathfrak{Q}$ *is a monotone, then* $\mathfrak{Q}^*$ *is monotone; if* $\mathfrak{Q}$ *is increasing, then* $\mathfrak{Q}^*$ *is decreasing; if* $\mathfrak{Q}$ *is decreasing, then* $\mathfrak{Q}^*$ *is increasing.*

Proof: Assume λ is the membership function of $\mathfrak{Q}$. Then $\mathfrak{Q}^*$ has a membership function $\lambda(n - x)$. The theorem follows from this fact immediately.

10.3 Uncertain Subject

Sometimes, we are interested in a subset of the universe of individuals, for example, "warm days", "young students" and "tall sportsmen". This section will model them by the concept of uncertain subject.

Definition 10.6 *(Liu [130]) Uncertain subject is an uncertain set containing some specified individuals in the universe.*

Example 10.22: "Warm days are here again" is a statement in which "warm days" is an uncertain subject that is an uncertain set on the universe of "all days", whose membership function may be defined by

$$\nu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/3, & \text{if } 15 \le x \le 18 \\ 1, & \text{if } 18 \le x \le 24 \\ (28-x)/4, & \text{if } 24 \le x \le 28 \\ 0, & \text{if } 28 \le x. \end{cases} \tag{10.39}$$

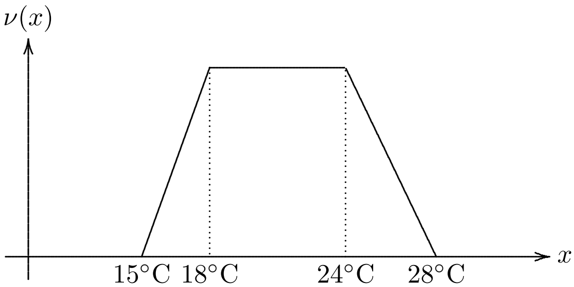


Figure 10.9: Membership Function of Subject "warm days"

Example 10.23: "Young students are tall" is a statement in which "young students" is an uncertain subject that is an uncertain set on the universe of "all students", whose membership function may be defined by

$$\nu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/5, & \text{if } 15 \le x \le 20 \\ 1, & \text{if } 20 \le x \le 35 \\ (45-x)/10, & \text{if } 35 \le x \le 45 \\ 0, & \text{if } x \ge 45. \end{cases} \tag{10.40}$$

Example 10.24: "Tall students are heavy" is a statement in which "tall students" is an uncertain subject that is an uncertain set on the universe of "all students", whose membership function may be defined by

$$\nu(x) = \begin{cases} 0, & \text{if } x \le 180 \\ (x-180)/5, & \text{if } 180 \le x \le 185 \\ 1, & \text{if } 185 \le x \le 195 \\ (200-x)/5, & \text{if } 195 \le x \le 200 \\ 0, & \text{if } x \ge 200. \end{cases} \tag{10.41}$$

Let S be an uncertain subject with membership function ν on the universe $A = \{a_1, a_2, \cdots, a_n\}$ of individuals. Then S is an uncertain set of A such

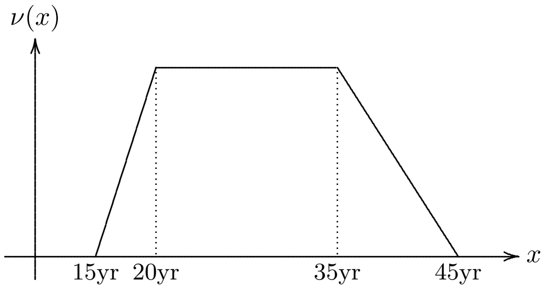


Figure 10.10: Membership Function of Subject "young students"

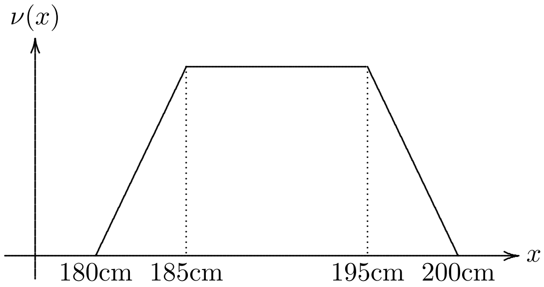


Figure 10.11: Membership Function of Subject "tall students"

that

$$\mathcal{M}\{a_i \in S\} = \nu(a_i), \quad i = 1, 2, \cdots, n. \tag{10.42}$$

In many cases, we are interested in some individuals a's with $\nu(a) \geq \omega$, where ω is a confidence level. Thus we have a subuniverse,

$$S_\omega = \{a \in A \,|\, \nu(a) \geq \omega\} \tag{10.43}$$

that will play a new universe of individuals we are talking about, and the individuals out of S_ω will be ignored at the confidence level ω.

Theorem 10.7 *Let ω_1 and ω_2 be confidence levels with $\omega_1 > \omega_2$, and let S_{ω_1} and S_{ω_2} be subuniverses with confidence levels ω_1 an ω_2, respectively. Then*

$$S_{\omega_1} \subset S_{\omega_2}. \tag{10.44}$$

That is, S_ω is a decreasing sequence of sets with respect to ω.

Proof: If $a \in S_{\omega_1}$, then $\nu(a) \geq \omega_1 > \omega_2$. Thus $a \in S_{\omega_2}$. It follows that $S_{\omega_1} \subset S_{\omega_2}$. Note that S_{ω_1} and S_{ω_2} may be empty.

10.4 Uncertain Predicate

There are numerous imprecise predicates in human language, for example, *warm*, *cold*, *hot*, *young*, *old*, *tall*, *small*, and *big*. This section will model them by the concept of uncertain predicate.

Definition 10.7 *(Liu [130]) Uncertain predicate is an uncertain set representing a property that the individuals have in common.*

Example 10.25: "Today is warm" is a statement in which "warm" is an uncertain predicate that may be represented by a membership function

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/3, & \text{if } 15 \le x \le 18 \\ 1, & \text{if } 18 \le x \le 24 \\ (28-x)/4, & \text{if } 24 \le x \le 28 \\ 0, & \text{if } 28 \le x. \end{cases} \tag{10.45}$$

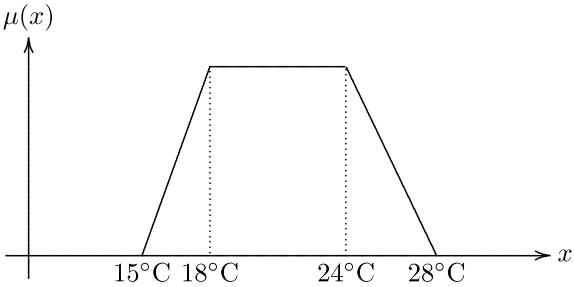


Figure 10.12: Membership Function of Predicate "*warm*"

Example 10.26: "John is young" is a statement in which "young" is an uncertain predicate that may be represented by a membership function

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/5, & \text{if } 15 \le x \le 20 \\ 1, & \text{if } 20 \le x \le 35 \\ (45-x)/10, & \text{if } 35 \le x \le 45 \\ 0, & \text{if } x \ge 45. \end{cases} \tag{10.46}$$

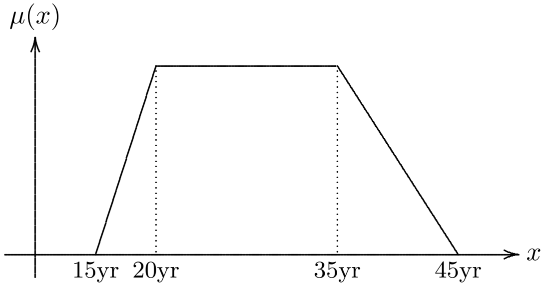


Figure 10.13: Membership Function of Predicate "young"

Example 10.27: "Tom is tall" is a statement in which "tall" is an uncertain predicate that may be represented by a membership function

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 180 \\ (x-180)/5, & \text{if } 180 \le x \le 185 \\ 1, & \text{if } 185 \le x \le 195 \\ (200-x)/5, & \text{if } 195 \le x \le 200 \\ 0, & \text{if } x \ge 200. \end{cases} \tag{10.47}$$

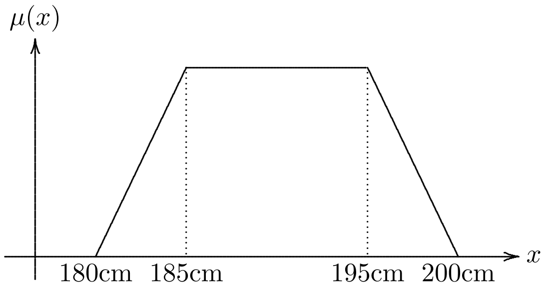


Figure 10.14: Membership Function of Predicate "tall"

Negated Predicate

Definition 10.8 *Let P be an uncertain predicate. Then its negated predicate $\neg P$ is the complement of P in the sense of uncertain set, i.e.,*

$$\neg P = P^c. \tag{10.48}$$

Theorem 10.8 *Let P be an uncertain predicate with membership function μ. Then its negated predicate $\neg P$ has a membership function*

$$\neg\mu(x) = 1 - \mu(x). \tag{10.49}$$

Proof: The theorem follows from the definition of negated predicate and the operational law of uncertain set immediately.

Example 10.28: Let P be the uncertain predicate "warm" defined by (10.45). Then its negated predicate $\neg P$ has a membership function

$$\neg\mu(x) = \begin{cases} 1, & \text{if } x \le 15 \\ (18-x)/3, & \text{if } 15 \le x \le 18 \\ 0, & \text{if } 18 \le x \le 24 \\ (x-24)/4, & \text{if } 24 \le x \le 28 \\ 1, & \text{if } 28 \le x. \end{cases} \tag{10.50}$$

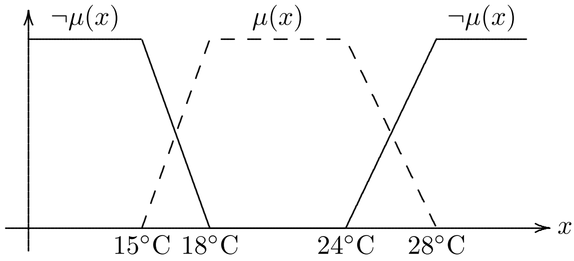


Figure 10.15: Membership Function of Negated Predicate of "*warm*"

Example 10.29: Let P be the negated predicate "*young*" defined by (10.46). Then its negated predicate $\neg P$ has a membership function

$$\neg\mu(x) = \begin{cases} 1, & \text{if } x \le 15 \\ (20-x)/5, & \text{if } 15 \le x \le 20 \\ 0, & \text{if } 20 \le x \le 35 \\ (x-35)/10, & \text{if } 35 \le x \le 45 \\ 1, & \text{if } x \ge 45. \end{cases} \tag{10.51}$$

Example 10.30: Let P be the uncertain predicate "*tall*" defined by (10.47). Then its negated predicate $\neg P$ has a membership function

$$\neg\mu(x) = \begin{cases} 1, & \text{if } x \le 180 \\ (185-x)/5, & \text{if } 180 \le x \le 185 \\ 0, & \text{if } 185 \le x \le 195 \\ (x-195)/5, & \text{if } 195 \le x \le 200 \\ 1, & \text{if } x \ge 200. \end{cases} \tag{10.52}$$

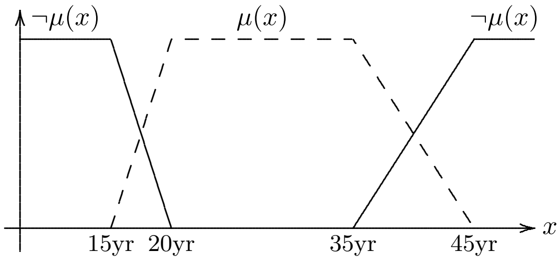


Figure 10.16: Membership Function of Negated Predicate of "*young*"

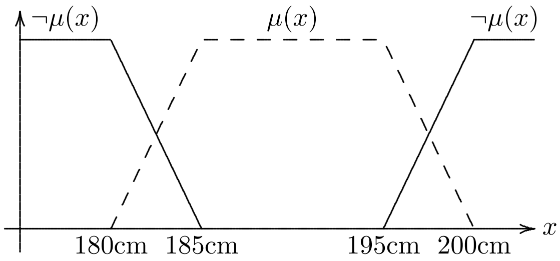


Figure 10.17: Membership Function of Negated Predicate of "*tall*"

Theorem 10.9 *Let P be an uncertain predicate. Then we have $\neg\neg P = P$.*

Proof: The theorem follows from $\neg\neg P = \neg P^c = (P^c)^c = P$.

10.5 Uncertain Proposition

Definition 10.9 *(Liu [130]) Assume that $\mathbb{Q}$ is an uncertain quantifier, S is an uncertain subject, and P is an uncertain predicate. Then the triplet*

$$(\mathbb{Q}, S, P) = \text{“}\mathbb{Q} \textit{ of } S \textit{ are } P\text{”} \tag{10.53}$$

is called an uncertain proposition.

Remark 10.2: Let A be the universe of individuals. Then $(\mathbb{Q}, A, P)$ is a special uncertain proposition because A itself is a special uncertain subject.

Remark 10.3: Let $\forall$ be the universal quantifier. Then $(\forall, A, P)$ is an uncertain proposition representing "all of A are P".

Remark 10.4: Let $\exists$ be the existential quantifier. Then $(\exists, A, P)$ is an uncertain proposition representing "at least one of A is P".

Example 10.31: "Almost all students are young" is an uncertain proposition in which the uncertain quantifier $\mathcal{Q}$ is "almost all", the uncertain subject S is "students" (the universe itself) and the uncertain predicate P is "young".

Example 10.32: "Most young students are tall" is an uncertain proposition in which the uncertain quantifier $\mathcal{Q}$ is "most", the uncertain subject S is "young students" and the uncertain predicate P is "tall".

Theorem 10.10 *(Liu [130], Logical Equivalence Theorem) Let* $(\mathcal{Q}, S, P)$ *be an uncertain proposition. Then*

$$(\mathcal{Q}^*, S, P) = (\mathcal{Q}, S, \neg P) \tag{10.54}$$

where $\mathcal{Q}^*$ *is the dual quantifier of* $\mathcal{Q}$ *and* $\neg P$ *is the negated predicate of* P.

Proof: Note that $(\mathcal{Q}^*, S, P)$ represents "$\mathcal{Q}^*$ of S are P". In fact, the statement "$\mathcal{Q}^*$ of S are P" implies "$\mathcal{Q}^{**}$ of S are not P". Since $\mathcal{Q}^{**} = \mathcal{Q}$, we obtain $(\mathcal{Q}, S, \neg P)$. Conversely, the statement "$\mathcal{Q}$ of S are not P" implies "$\mathcal{Q}^*$ of S are P", i.e., $(\mathcal{Q}^*, S, P)$. Thus (10.54) is verified.

Example 10.33: When $\mathcal{Q}^* = \neg\forall$, we have $\mathcal{Q} = \exists$. If $S = A$, then (10.54) becomes the classical equivalence

$$(\neg\forall, A, P) = (\exists, A, \neg P). \tag{10.55}$$

Example 10.34: When $\mathcal{Q}^* = \neg\exists$, we have $\mathcal{Q} = \forall$. If $S = A$, then (10.54) becomes the classical equivalence

$$(\neg\exists, A, P) = (\forall, A, \neg P). \tag{10.56}$$

10.6 Truth Value

Let $(\mathcal{Q}, S, P)$ be an uncertain proposition. The truth value of $(\mathcal{Q}, S, P)$ should be the uncertain measure that "$\mathcal{Q}$ of S are P". That is,

$$T(\mathcal{Q}, S, P) = \mathcal{M}\{\mathcal{Q} \text{ of } S \text{ are } P\}. \tag{10.57}$$

However, it is impossible for us to deduce the value of $\mathcal{M}\{\mathcal{Q} \text{ of } S \text{ are } P\}$ from the information of $\mathcal{Q}$, S and P within the framework of uncertain set theory. Thus we need an additional formula to compose $\mathcal{Q}$, S and P.

Definition 10.10 *(Liu [130]) Let* $(\mathcal{Q}, S, P)$ *be an uncertain proposition in which* $\mathcal{Q}$ *is a unimodal uncertain quantifier with membership function* λ, S *is an uncertain subject with membership function* ν, *and* P *is an uncertain predicate with membership function* μ. *Then the truth value of* $(\mathcal{Q}, S, P)$ *with respect to the universe* A *is*

$$T(\mathcal{Q}, S, P) = \sup_{0\le\omega\le 1}\left(\omega \wedge \sup_{K\in\mathbb{K}_\omega}\inf_{a\in K}\mu(a) \wedge \sup_{K\in\mathbb{K}_\omega^*}\inf_{a\in K}\neg\mu(a)\right) \tag{10.58}$$

where

$$\mathbb{K}_\omega = \{K \subset S_\omega \,|\, \lambda(|K|) \geq \omega\}\,, \tag{10.59}$$

$$\mathbb{K}_\omega^* = \{K \subset S_\omega \,|\, \lambda(|S_\omega| - |K|) \geq \omega\}\,, \tag{10.60}$$

$$S_\omega = \{a \in A \,|\, \nu(a) \geq \omega\}\,. \tag{10.61}$$

Remark 10.5: Keep in mind that the truth value formula (10.58) is vacuous if the individual feature data of the universe A are not available.

Remark 10.6: The symbol $|K|$ represents the cardinality of the set K. For example, $|\emptyset| = 0$ and $|\{2,5,6\}| = 3$.

Remark 10.7: Note that $\neg\mu$ is the membership function of the negated predicate of P, and

$$\neg\mu(a) = 1 - \mu(a). \tag{10.62}$$

Remark 10.8: When the subset K of individuals becomes an empty set $\emptyset$, we will define

$$\inf_{a\in\emptyset} \mu(a) = \inf_{a\in\emptyset} \neg\mu(a) = 1. \tag{10.63}$$

Remark 10.9: If $\mathcal{Q}$ is an uncertain percentage rather than an absolute quantity, then $\mathbb{K}_\omega$ and $\mathbb{K}_\omega^*$ are defined by

$$\mathbb{K}_\omega = \left\{K \subset S_\omega \;\middle|\; \lambda\left(\frac{|K|}{|S_\omega|}\right) \geq \omega\right\}, \tag{10.64}$$

$$\mathbb{K}_\omega^* = \left\{K \subset S_\omega \;\middle|\; \lambda\left(1 - \frac{|K|}{|S_\omega|}\right) \geq \omega\right\}. \tag{10.65}$$

Remark 10.10: If the uncertain subject S degenerates to the universe A, then the truth value of $(\mathcal{Q}, A, P)$ is

$$T(\mathcal{Q}, A, P) = \sup_{0\leq\omega\leq 1}\left(\omega \wedge \sup_{K\in\mathbb{K}_\omega}\inf_{a\in K}\mu(a) \wedge \sup_{K\in\mathbb{K}_\omega^*}\inf_{a\in K}\neg\mu(a)\right) \tag{10.66}$$

where

$$\mathbb{K}_\omega = \{K \subset A \,|\, \lambda(|K|) \geq \omega\}\,, \tag{10.67}$$

$$\mathbb{K}_\omega^* = \{K \subset A \,|\, \lambda(|A| - |K|) \geq \omega\}\,. \tag{10.68}$$

Exercise 10.1: If the uncertain quantifier $\mathcal{Q} = \forall$ and the uncertain subject $S = A$, then for any $\omega > 0$, we have

$$\mathbb{K}_\omega = \{A\}, \quad \mathbb{K}_\omega^* = \{\emptyset\}. \tag{10.69}$$

Show that

$$T(\forall, A, P) = \inf_{a \in A} \mu(a). \tag{10.70}$$

Exercise 10.2: If the uncertain quantifier $\mathcal{Q} = \exists$ and the uncertain subject $S = A$, then for any $\omega > 0$, we have

$$\mathbb{K}_\omega = \{\text{any nonempty subsets of } A\}, \tag{10.71}$$

$$\mathbb{K}_\omega^* = \{\text{any proper subsets of } A\}. \tag{10.72}$$

Note that $\mathbb{K}_\omega$ contains A but $\mathbb{K}_\omega^*$ does not. Show that

$$T(\exists, A, P) = \sup_{a \in A} \mu(a). \tag{10.73}$$

Exercise 10.3: If the uncertain quantifier $\mathcal{Q} = \neg\forall$ and the uncertain subject $S = A$, then for any $\omega > 0$, we have

$$\mathbb{K}_\omega = \{\text{any proper subsets of } A\}, \tag{10.74}$$

$$\mathbb{K}_\omega^* = \{\text{any nonempty subsets of } A\}. \tag{10.75}$$

Show that

$$T(\neg\forall, A, P) = 1 - \inf_{a \in A} \mu(a). \tag{10.76}$$

Exercise 10.4: If the uncertain quantifier $\mathcal{Q} = \neg\exists$ and the uncertain subject $S = A$, then for any $\omega > 0$, we have

$$\mathbb{K}_\omega = \{\emptyset\}, \quad \mathbb{K}_\omega^* = \{A\}. \tag{10.77}$$

Show that

$$T(\neg\exists, A, P) = 1 - \sup_{a \in A} \mu(a). \tag{10.78}$$

Theorem 10.11 *(Liu [130], Truth Value Theorem) Let $(\mathcal{Q}, S, P)$ be an uncertain proposition in which $\mathcal{Q}$ is a unimodal uncertain quantifier with membership function λ, S is an uncertain subject with membership function ν, and P is an uncertain predicate with membership function μ. Then the truth value of $(\mathcal{Q}, S, P)$ is*

$$T(\mathcal{Q}, S, P) = \sup_{0 \le \omega \le 1} (\omega \wedge \Delta(k_\omega) \wedge \Delta^*(k_\omega^*)) \tag{10.79}$$

where

$$k_\omega = \min\{x \mid \lambda(x) \ge \omega\}, \tag{10.80}$$

$$\Delta(k_\omega) = \text{the } k_\omega\text{-th largest value of } \{\mu(a_i) \mid a_i \in S_\omega\}, \tag{10.81}$$

$$k_\omega^* = |S_\omega| - \max\{x \mid \lambda(x) \ge \omega\}, \tag{10.82}$$

$$\Delta^*(k_\omega^*) = \text{the } k_\omega^*\text{-th largest value of } \{1 - \mu(a_i) \mid a_i \in S_\omega\}. \tag{10.83}$$

Proof: Since the supremum is achieved at the subset with minimum cardinality, we have

$$\sup_{K\in\mathbb{K}_\omega}\inf_{a\in K}\mu(a)=\sup_{K\subset S_\omega,|K|=k_\omega}\inf_{a\in K}\mu(a)=\Delta(k_\omega),$$

$$\sup_{K\in\mathbb{K}^*_\omega}\inf_{a\in K}\neg\mu(a)=\sup_{K\subset S_\omega,|K|=k^*_\omega}\inf_{a\in K}\neg\mu(a)=\Delta^*(k^*_\omega).$$

The theorem is thus verified. Please note that $\Delta(0)=\Delta^*(0)=1$.

Remark 10.11: If $\mathcal{Q}$ is an uncertain percentage, then k_ω and k^*_ω are defined by

$$k_\omega=\min\left\{x \;\middle|\; \lambda\left(\frac{x}{|S_\omega|}\right)\ge\omega\right\}, \tag{10.84}$$

$$k^*_\omega=|S_\omega|-\max\left\{x \;\middle|\; \lambda\left(\frac{x}{|S_\omega|}\right)\ge\omega\right\}. \tag{10.85}$$

Remark 10.12: If the uncertain subject S degenerates to the universe of individuals $A=\{a_1,a_2,\cdots,a_n\}$, then the truth value of $(\mathcal{Q},A,P)$ is

$$T(\mathcal{Q},A,P)=\sup_{0\le\omega\le1}\left(\omega\wedge\Delta(k_\omega)\wedge\Delta^*(k^*_\omega)\right) \tag{10.86}$$

where

$$k_\omega=\min\left\{x\,|\,\lambda(x)\ge\omega\right\}, \tag{10.87}$$

$$\Delta(k_\omega)=\text{the } k_\omega\text{-th largest value of } \mu(a_1),\mu(a_2),\cdots,\mu(a_n), \tag{10.88}$$

$$k^*_\omega=n-\max\{x\,|\,\lambda(x)\ge\omega\}, \tag{10.89}$$

$$\Delta^*(k^*_\omega)=\text{the } k^*_\omega\text{-th largest value of } 1-\mu(a_1),\cdots,1-\mu(a_n). \tag{10.90}$$

Exercise 10.5: If the uncertain quantifier $\mathcal{Q}=\{m,m+1,\cdots,n\}$ (i.e., "*there exist at least* m") with $m\ge1$, then we have $k_\omega=m$ and $k^*_\omega=0$. Show that

$$T(\mathcal{Q},A,P)=\text{the } m\text{th largest value of } \mu(a_1),\mu(a_2),\cdots,\mu(a_n). \tag{10.91}$$

Exercise 10.6: If the uncertain quantifier $\mathcal{Q}=\{0,1,2,\ldots,m\}$ (i.e., "*there exist at most* m") with $m<n$, then we have $k_\omega=0$ and $k^*_\omega=n-m$. Show that

$$T(\mathcal{Q},A,P)=\text{the } (n-m)\text{th largest value of } 1-\mu(a_1),1-\mu(a_2),\cdots,1-\mu(a_n).$$

10.7 Algorithm

In order to calculate $T(\mathbb{Q}, S, P)$ based on the truth value formula (10.58), a truth value algorithm is given as follows:

Step 1. Set $\omega = 1$ and $\varepsilon = 0.01$ (a predetermined precision).

Step 2. Calculate $S_\omega = \{a \in A \mid \nu(a) \geq \omega\}$ and $k = \min\{x \mid \lambda(x) \geq \omega\}$ as well as $k^* = |S_\omega| - \max\{x \mid \lambda(x) \geq \omega\}$.

Step 3. If $\Delta(k) \wedge \Delta^*(k^*) < \omega$, then $\omega \leftarrow \omega - \varepsilon$ and go to Step 2. Otherwise, output the truth value ω and stop.

Remark 10.13: If $\mathbb{Q}$ is an uncertain percentage, then k_ω and k^*_ω in the truth value algorithm are replaced with (10.84) and (10.85), respectively.

Example 10.35: Assume that the daily temperatures of some week from Monday to Sunday are

$$22,\ 23,\ 25,\ 28,\ 30,\ 32,\ 36 \tag{10.92}$$

in centigrades, respectively. Consider an uncertain proposition

$$(\mathbb{Q}, A, P) = \text{“two or three days are warm”}. \tag{10.93}$$

Note that the uncertain quantifier is $\mathbb{Q} = \{2, 3\}$. We also suppose the uncertain predicate $P =$ “warm” has a membership function

$$\mu(x) = \begin{cases} 0, & \text{if } x \leq 15 \\ (x-15)/3, & \text{if } 15 \leq x \leq 18 \\ 1, & \text{if } 18 \leq x \leq 24 \\ (28-x)/4, & \text{if } 24 \leq x \leq 28 \\ 0, & \text{if } 28 \leq x. \end{cases} \tag{10.94}$$

It is clear that Monday and Tuesday are warm with truth value 1, and Wednesday is warm with truth value 0.75. But Thursday to Sunday are not “warm” at all (in fact, they are “hot”). Intuitively, the uncertain proposition “two or three days are warm” should be completely true. The truth value algorithm (http://orsc.edu.cn/liu/resources.htm) yields that the truth value is

$$T(\text{“two or three days are warm”}) = 1. \tag{10.95}$$

This is an intuitively expected result. In addition, we also have

$$T(\text{“two days are warm”}) = 0.25, \tag{10.96}$$

$$T(\text{“three days are warm”}) = 0.75. \tag{10.97}$$

Example 10.36: Assume that in a class there are 15 students whose ages are

$$21, 22, 22, 23, 24, 25, 26, 27, 28, 30, 32, 35, 36, 38, 40 \tag{10.98}$$

in years. Consider an uncertain proposition

$$(\mathcal{Q}, A, P) = \text{“almost all students are young”}. \tag{10.99}$$

Suppose the uncertain quantifier $\mathcal{Q}$ = “almost all” has a membership function

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \le x \le 10 \\ (x-10)/3, & \text{if } 10 \le x \le 13 \\ 1, & \text{if } 13 \le x \le 15, \end{cases} \tag{10.100}$$

and the uncertain predicate P = “young” has a membership function

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/5, & \text{if } 15 \le x \le 20 \\ 1, & \text{if } 20 \le x \le 35 \\ (45-x)/10, & \text{if } 35 \le x \le 45 \\ 0, & \text{if } x \ge 45. \end{cases} \tag{10.101}$$

The truth value algorithm (http://orsc.edu.cn/liu/resources.htm) yields that the uncertain proposition has a truth value

$$T(\text{“almost all students are young”}) = 0.9. \tag{10.102}$$

Example 10.37: Assume that in a team there are 16 sportsmen whose heights are

$$\begin{array}{c} 175, 178, 178, 180, 183, 184, 186, 186 \\ 188, 190, 192, 192, 193, 194, 195, 196 \end{array} \tag{10.103}$$

in centimeters. Consider an uncertain proposition

$$(\mathcal{Q}, A, P) = \text{“about 70\% of sportsmen are tall”}. \tag{10.104}$$

Suppose the uncertain quantifier $\mathcal{Q}$ = “about 70%” has a membership function

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \le x \le 0.6 \\ 20(x-0.6), & \text{if } 0.6 \le x \le 0.65 \\ 1, & \text{if } 0.65 \le x \le 0.75 \\ 20(0.8-x), & \text{if } 0.75 \le x \le 0.8 \\ 0, & \text{if } 0.8 \le x \le 1 \end{cases} \tag{10.105}$$

and the uncertain predicate $P =$ "tall" has a membership function

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 180 \\ (x-180)/5, & \text{if } 180 \le x \le 185 \\ 1, & \text{if } 185 \le x \le 195 \\ (200-x)/5, & \text{if } 195 \le x \le 200 \\ 0, & \text{if } x \ge 200. \end{cases} \tag{10.106}$$

The truth value algorithm (http://orsc.edu.cn/liu/resources.htm) yields that the uncertain proposition has a truth value

$$T(\text{"about 70\% of sportsmen are tall"}) = 0.8. \tag{10.107}$$

Example 10.38: Assume that in a class there are 18 students whose ages and heights are

$$\begin{array}{l} (24,185),\ (25,190),\ (26,184),\ (26,170),\ (27,187),\ (27,188) \\ (28,160),\ (30,190),\ (32,185),\ (33,176),\ (35,185),\ (36,188) \\ (38,164),\ (38,178),\ (39,182),\ (40,186),\ (42,165),\ (44,170) \end{array} \tag{10.108}$$

in years and centimeters. Consider an uncertain proposition

$$(\mathcal{Q}, S, P) = \text{"most young students are tall"}. \tag{10.109}$$

Suppose the uncertain quantifier (percentage) $\mathcal{Q} =$ "most" has a membership function

$$\lambda(x) = \begin{cases} 0, & \text{if } 0 \le x \le 0.7 \\ 20(x-0.7), & \text{if } 0.7 \le x \le 0.75 \\ 1, & \text{if } 0.75 \le x \le 0.85 \\ 20(0.9-x), & \text{if } 0.85 \le x \le 0.9 \\ 0, & \text{if } 0.9 \le x \le 1. \end{cases} \tag{10.110}$$

Note that each individual is described by a feature data (y, z), where y represents ages and z represents heights. In this case, the uncertain subject $S =$ "young students" has a membership function

$$\nu(y) = \begin{cases} 0, & \text{if } y \le 15 \\ (y-15)/5, & \text{if } 15 \le y \le 20 \\ 1, & \text{if } 20 \le y \le 35 \\ (45-y)/10, & \text{if } 35 \le y \le 45 \\ 0, & \text{if } y \ge 45 \end{cases} \tag{10.111}$$

and the uncertain predicate $P =$ "tall" has a membership function

$$\mu(z) = \begin{cases} 0, & \text{if } z \le 180 \\ (z-180)/5, & \text{if } 180 \le z \le 185 \\ 1, & \text{if } 185 \le z \le 195 \\ (200-z)/5, & \text{if } 195 \le z \le 200 \\ 0, & \text{if } z \ge 200. \end{cases} \tag{10.112}$$

The truth value algorithm yields that the uncertain proposition has a truth value

$$T(\text{"most young students are tall"}) = 0.8. \tag{10.113}$$

10.8 Linguistic Summarizer

Linguistic summary is a human language statement that is concise and easy-to-understand by humans. For example, "most young students are tall" is a linguistic summary of students' ages and heights. Thus a linguistic summary is a special uncertain proposition whose uncertain quantifier, uncertain subject and uncertain predicate are linguistic terms. Uncertain logic provides a flexible means that is capable of extracting linguistic summary from a collection of raw data.

What inputs does the uncertain logic need? First, we should have some raw data (i.e., the individual feature data),

$$A = \{a_1, a_2, \cdots, a_n\}. \tag{10.114}$$

Next, we should have some linguistic terms to represent quantifiers, for example, "most" and "all". Denote them by a collection of uncertain quantifiers,

$$\mathbb{Q} = \{\mathcal{Q}_1, \mathcal{Q}_2, \cdots, \mathcal{Q}_m\}. \tag{10.115}$$

Then, we should have some linguistic terms to represent subjects, for example, "young students" and "old students". Denote them by a collection of uncertain subjects,

$$\mathbb{S} = \{S_1, S_2, \cdots, S_n\}. \tag{10.116}$$

Last, we should have some linguistic terms to represent predicates, for example, "short" and "tall". Denote them by a collection of uncertain predicates,

$$\mathbb{P} = \{P_1, P_2, \cdots, P_k\}. \tag{10.117}$$

One problem of data mining is to choose an uncertain quantifier $\mathcal{Q} \in \mathbb{Q}$, an uncertain subject $S \in \mathbb{S}$ and an uncertain predicate $P \in \mathbb{P}$ such that the truth value of the linguistic summary "$\mathcal{Q}$ of S are P" to be extracted is at least β, i.e.,

$$T(\mathcal{Q}, S, P) \ge \beta \tag{10.118}$$

for the universe $A = \{a_1, a_2, \cdots, a_n\}$, where β is a confidence level. In order to solve this problem, Liu [130] proposed the following linguistic summarizer,

$$\left\{\begin{array}{l}\text{Find } \mathcal{Q},\, S \text{ and } P\\ \text{subject to:}\\ \qquad \mathcal{Q} \in \mathbb{Q}\\ \qquad S \in \mathbb{S}\\ \qquad P \in \mathbb{P}\\ \qquad T(\mathcal{Q}, S, P) \geq \beta.\end{array}\right. \tag{10.119}$$

Each solution $(\overline{\mathcal{Q}}, \overline{S}, \overline{P})$ of the linguistic summarizer (10.119) produces a linguistic summary "$\overline{\mathcal{Q}}$ of $\overline{S}$ are $\overline{P}$".

Example 10.39: Assume that in a class there are 18 students whose ages and heights are

$$\begin{array}{l}(24,185),\ (25,190),\ (26,184),\ (26,170),\ (27,187),\ (27,188)\\ (28,160),\ (30,190),\ (32,185),\ (33,176),\ (35,185),\ (36,188)\\ (38,164),\ (38,178),\ (39,182),\ (40,186),\ (42,165),\ (44,170)\end{array} \tag{10.120}$$

in years and centimeters. Suppose we have three linguistic terms "about half", "most" and "all" as uncertain quantifiers whose membership functions are

$$\lambda_{half}(x) = \left\{\begin{array}{cl} 0, & \text{if } 0 \leq x \leq 0.4\\ 20(x-0.4), & \text{if } 0.4 \leq x \leq 0.45\\ 1, & \text{if } 0.45 \leq x \leq 0.55\\ 20(0.6-x), & \text{if } 0.55 \leq x \leq 0.6\\ 0, & \text{if } 0.6 \leq x \leq 1,\end{array}\right. \tag{10.121}$$

$$\lambda_{most}(x) = \left\{\begin{array}{cl} 0, & \text{if } 0 \leq x \leq 0.7\\ 20(x-0.7), & \text{if } 0.7 \leq x \leq 0.75\\ 1, & \text{if } 0.75 \leq x \leq 0.85\\ 20(0.9-x), & \text{if } 0.85 \leq x \leq 0.9\\ 0, & \text{if } 0.9 \leq x \leq 1,\end{array}\right. \tag{10.122}$$

$$\lambda_{all}(x) = \left\{\begin{array}{ll} 1, & \text{if } x = 1\\ 0, & \text{if } 0 \leq x < 1,\end{array}\right. \tag{10.123}$$

respectively. Denote the collection of uncertain quantifiers by

$$\mathbb{Q} = \{\text{“about half”},\ \text{“most”}, \text{“all”}\}. \tag{10.124}$$

We also have three linguistic terms "young students", "middle-aged students" and "old students" as uncertain subjects whose membership functions are

$$\nu_{young}(y) = \begin{cases} 0, & \text{if } y \le 15 \\ (y-15)/5, & \text{if } 15 \le y \le 20 \\ 1, & \text{if } 20 \le y \le 35 \\ (45-y)/10, & \text{if } 35 \le y \le 45 \\ 0, & \text{if } y \ge 45, \end{cases} \tag{10.125}$$

$$\nu_{middle}(y) = \begin{cases} 0, & \text{if } y \le 40 \\ (y-40)/5, & \text{if } 40 \le y \le 45 \\ 1, & \text{if } 45 \le y \le 55 \\ (60-y)/5, & \text{if } 55 \le y \le 60 \\ 0, & \text{if } y \ge 60, \end{cases} \tag{10.126}$$

$$\nu_{old}(y) = \begin{cases} 0, & \text{if } y \le 55 \\ (y-55)/5, & \text{if } 55 \le y \le 60 \\ 1, & \text{if } 60 \le y \le 80 \\ (85-y)/5, & \text{if } 80 \le y \le 85 \\ 1, & \text{if } y \ge 85, \end{cases} \tag{10.127}$$

respectively. Denote the collection of uncertain subjects by

$$\mathbb{S} = \{\text{"young students", "middle-aged students", "old students"}\}. \tag{10.128}$$

Finally, we suppose that there are two linguistic terms "short" and "tall" as uncertain predicates whose membership functions are

$$\mu_{short}(z) = \begin{cases} 0, & \text{if } z \le 145 \\ (z-145)/5, & \text{if } 145 \le z \le 150 \\ 1, & \text{if } 150 \le z \le 155 \\ (160-z)/5, & \text{if } 155 \le z \le 160 \\ 0, & \text{if } z \ge 200, \end{cases} \tag{10.129}$$

$$\mu_{tall}(z) = \begin{cases} 0, & \text{if } z \le 180 \\ (z-180)/5, & \text{if } 180 \le z \le 185 \\ 1, & \text{if } 185 \le z \le 195 \\ (200-z)/5, & \text{if } 195 \le z \le 200 \\ 0, & \text{if } z \ge 200, \end{cases} \tag{10.130}$$

respectively. Denote the collection of uncertain predicates by

$$\mathbb{P} = \{\text{"short", "tall"}\}. \tag{10.131}$$

We would like to extract an uncertain quantifier $\mathcal{Q} \in \mathbb{Q}$, an uncertain subject $S \in \mathbb{S}$ and an uncertain predicate $P \in \mathbb{P}$ such that the truth value of the linguistic summary "$\mathcal{Q}$ of S are P" to be extracted is at least 0.8, i.e.,

$$T(\mathcal{Q}, S, P) \geq 0.8 \tag{10.132}$$

where 0.8 is a predetermined confidence level. The linguistic summarizer (10.119) yields

$$\overline{\mathcal{Q}} = \text{"most"}, \quad \overline{S} = \text{"young students"}, \quad \overline{P} = \text{"tall"}$$

and then extracts a linguistic summary "most young students are tall".

10.9 Bibliographic Notes

Based on uncertain set theory, uncertain logic was designed by Liu [130] in 2011 for dealing with human language by using the truth value formula for uncertain propositions. As an application of uncertain logic, Liu [130] also proposed a linguistic summarizer that provides a means for extracting linguistic summary from a collection of raw data.

Chapter 11

Uncertain Inference

Uncertain inference is a process of deriving consequences from human knowledge via uncertain set theory. This chapter will introduce a family of uncertain inference rules, uncertain system, and uncertain control with application to an inverted pendulum system.

11.1 Uncertain Inference Rule

Let $\mathbb{X}$ and $\mathbb{Y}$ be two concepts. It is assumed that we only have a single if-then rule,

$$\text{“if } \mathbb{X} \text{ is } \xi \text{ then } \mathbb{Y} \text{ is } \eta\text{”} \tag{11.1}$$

where ξ and η are two uncertain sets. We first introduce the following inference rule.

Inference Rule 11.1 *(Liu [127]) Let $\mathbb{X}$ and $\mathbb{Y}$ be two concepts. Assume a rule "if $\mathbb{X}$ is an uncertain set ξ then $\mathbb{Y}$ is an uncertain set η". From $\mathbb{X}$ is a constant a we infer that $\mathbb{Y}$ is an uncertain set*

$$\eta^* = \eta|_{a\in\xi} \tag{11.2}$$

which is the conditional uncertain set of η given $a \in \xi$. The inference rule is represented by

$$\begin{array}{l} \text{Rule: If } \mathbb{X} \text{ is } \xi \text{ then } \mathbb{Y} \text{ is } \eta \\ \text{From: } \mathbb{X} \text{ is a constant } a \\ \hline \text{Infer: } \mathbb{Y} \text{ is } \eta^* = \eta|_{a\in\xi} \end{array} \tag{11.3}$$

Theorem 11.1 *Let ξ and η be independent uncertain sets with membership functions μ and ν, respectively. If ξ^* is a constant a, then the inference rule*

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5_12

11.1 yields that η^* has a membership function

$$\nu^*(y) = \begin{cases} \dfrac{\nu(y)}{\mu(a)}, & \textit{if } \nu(y) < \mu(a)/2 \\ \dfrac{\nu(y)+\mu(a)-1}{\mu(a)}, & \textit{if } \nu(y) > 1-\mu(a)/2 \\ 0.5, & \textit{otherwise.} \end{cases} \tag{11.4}$$

Proof: It follows from the inference rule 11.1 that η^* has a membership function

$$\nu^*(y) = \mathcal{M}\{y \in \eta | a \in \xi\}.$$

By using the definition of conditional uncertainty, we have

$$\mathcal{M}\{y \in \eta | a \in \xi\} = \begin{cases} \dfrac{\mathcal{M}\{y \in \eta\}}{\mathcal{M}\{a \in \xi\}}, & \text{if } \dfrac{\mathcal{M}\{y \in \eta\}}{\mathcal{M}\{a \in \xi\}} < 0.5 \\ 1 - \dfrac{\mathcal{M}\{y \notin \eta\}}{\mathcal{M}\{a \in \xi\}}, & \text{if } \dfrac{\mathcal{M}\{y \notin \eta\}}{\mathcal{M}\{a \in \xi\}} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

The equation (11.4) follows from $\mathcal{M}\{y \in \eta\} = \nu(y)$, $\mathcal{M}\{y \notin \eta\} = 1 - \nu(y)$ and $\mathcal{M}\{a \in \xi\} = \mu(a)$ immediately. The theorem is proved.

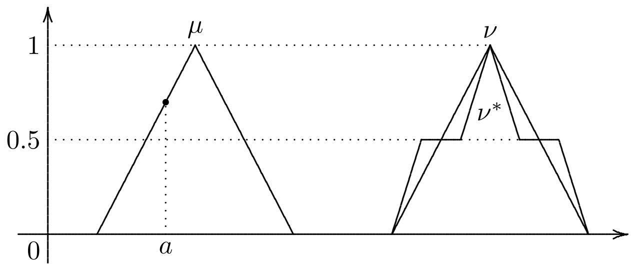


Figure 11.1: Graphical Illustration of Inference Rule. Reprinted from Liu [129].

Inference Rule 11.2 *(Gao, Gao and Ralescu [49]) Let $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{Z}$ be three concepts. Assume a rule "if $\mathbb{X}$ is an uncertain set ξ and $\mathbb{Y}$ is an uncertain set η then $\mathbb{Z}$ is an uncertain set τ". From $\mathbb{X}$ is a constant a and $\mathbb{Y}$ is a constant b we infer that $\mathbb{Z}$ is an uncertain set*

$$\tau^* = \tau|_{(a\in\xi)\cap(b\in\eta)} \tag{11.5}$$

which is the conditional uncertain set of τ given $a \in \xi$ and $b \in \eta$. The inference rule is represented by

$$\begin{array}{l} \text{Rule: If } \mathbb{X} \text{ is } \xi \text{ and } \mathbb{Y} \text{ is } \eta \text{ then } \mathbb{Z} \text{ is } \tau \\ \text{From: } \mathbb{X} \text{ is } a \text{ and } \mathbb{Y} \text{ is } b \\ \hline \text{Infer: } \mathbb{Z} \text{ is } \tau^* = \tau|_{(a\in\xi)\cap(b\in\eta)} \end{array} \tag{11.6}$$

Theorem 11.2 *Let ξ, η, τ be independent uncertain sets with membership functions μ, ν, λ, respectively. If ξ^* is a constant a and η^* is a constant b, then the inference rule 11.2 yields that τ^* has a membership function*

$$\lambda^*(z) = \begin{cases} \dfrac{\lambda(z)}{\mu(a) \wedge \nu(b)}, & \text{if } \lambda(z) < \dfrac{\mu(a) \wedge \nu(b)}{2} \\ \dfrac{\lambda(z) + \mu(a) \wedge \nu(b) - 1}{\mu(a) \wedge \nu(b)}, & \text{if } \lambda(z) > 1 - \dfrac{\mu(a) \wedge \nu(b)}{2} \\ 0.5, & \text{otherwise.} \end{cases} \tag{11.7}$$

Proof: It follows from the inference rule 11.2 that τ^* has a membership function

$$\lambda^*(z) = \mathcal{M}\{z \in \tau | (a \in \xi) \cap (b \in \eta)\}.$$

By using the definition of conditional uncertainty, $\mathcal{M}\{z \in \tau | (a \in \xi) \cap (b \in \eta)\}$ is

$$\begin{cases} \dfrac{\mathcal{M}\{z \in \tau\}}{\mathcal{M}\{(a \in \xi) \cap (b \in \eta)\}}, & \text{if } \dfrac{\mathcal{M}\{z \in \tau\}}{\mathcal{M}\{(a \in \xi) \cap (b \in \eta)\}} < 0.5 \\ 1 - \dfrac{\mathcal{M}\{z \notin \tau\}}{\mathcal{M}\{(a \in \xi) \cap (b \in \eta)\}}, & \text{if } \dfrac{\mathcal{M}\{z \notin \tau\}}{\mathcal{M}\{(a \in \xi) \cap (b \in \eta)\}} < 0.5 \\ 0.5, & \text{otherwise.} \end{cases}$$

The theorem follows from $\mathcal{M}\{z \in \tau\} = \lambda(z)$, $\mathcal{M}\{z \notin \tau\} = 1 - \lambda(z)$ and $\mathcal{M}\{(a \in \xi) \cap (b \in \eta)\} = \mu(a) \wedge \nu(b)$ immediately.

Inference Rule 11.3 *(Gao, Gao and Ralescu [49]) Let $\mathbb{X}$ and $\mathbb{Y}$ be two concepts. Assume two rules "if $\mathbb{X}$ is an uncertain set ξ_1 then $\mathbb{Y}$ is an uncertain set η_1" and "if $\mathbb{X}$ is an uncertain set ξ_2 then $\mathbb{Y}$ is an uncertain set η_2". From $\mathbb{X}$ is a constant a we infer that $\mathbb{Y}$ is an uncertain set*

$$\eta^* = \frac{\mathcal{M}\{a \in \xi_1\} \cdot \eta_1|_{a \in \xi_1}}{\mathcal{M}\{a \in \xi_1\} + \mathcal{M}\{a \in \xi_2\}} + \frac{\mathcal{M}\{a \in \xi_2\} \cdot \eta_2|_{a \in \xi_2}}{\mathcal{M}\{a \in \xi_1\} + \mathcal{M}\{a \in \xi_2\}}. \tag{11.8}$$

The inference rule is represented by

$$\begin{array}{l} \text{Rule 1: If } \mathbb{X} \text{ is } \xi_1 \text{ then } \mathbb{Y} \text{ is } \eta_1 \\ \text{Rule 2: If } \mathbb{X} \text{ is } \xi_2 \text{ then } \mathbb{Y} \text{ is } \eta_2 \\ \text{From: } \mathbb{X} \text{ is a constant } a \\ \hline \text{Infer: } \mathbb{Y} \text{ is } \eta^* \text{ determined by (11.8)} \end{array} \tag{11.9}$$

Theorem 11.3 *Let $\xi_1, \xi_2, \eta_1, \eta_2$ be independent uncertain sets with membership functions $\mu_1, \mu_2, \nu_1, \nu_2$, respectively. If ξ^* is a constant a, then the inference rule 11.3 yields*

$$\eta^* = \frac{\mu_1(a)}{\mu_1(a) + \mu_2(a)} \eta_1^* + \frac{\mu_2(a)}{\mu_1(a) + \mu_2(a)} \eta_2^* \tag{11.10}$$

where η_1^* and η_2^* are uncertain sets whose membership functions are respectively given by

$$\nu_1^*(y) = \begin{cases} \dfrac{\nu_1(y)}{\mu_1(a)}, & \text{if } \nu_1(y) < \mu_1(a)/2 \\ \dfrac{\nu_1(y)+\mu_1(a)-1}{\mu_1(a)}, & \text{if } \nu_1(y) > 1-\mu_1(a)/2 \\ 0.5, & \text{otherwise,} \end{cases} \tag{11.11}$$

$$\nu_2^*(y) = \begin{cases} \dfrac{\nu_2(y)}{\mu_2(a)}, & \text{if } \nu_2(y) < \mu_2(a)/2 \\ \dfrac{\nu_2(y)+\mu_2(a)-1}{\mu_2(a)}, & \text{if } \nu_2(y) > 1-\mu_2(a)/2 \\ 0.5, & \text{otherwise.} \end{cases} \tag{11.12}$$

Proof: It follows from the inference rule 11.3 that the uncertain set η^* is just

$$\eta^* = \frac{\mathcal{M}\{a \in \xi_1\} \cdot \eta_1|_{a\in\xi_1}}{\mathcal{M}\{a \in \xi_1\} + \mathcal{M}\{a \in \xi_2\}} + \frac{\mathcal{M}\{a \in \xi_2\} \cdot \eta_2|_{a\in\xi_2}}{\mathcal{M}\{a \in \xi_1\} + \mathcal{M}\{a \in \xi_2\}}.$$

The theorem follows from $\mathcal{M}\{a \in \xi_1\} = \mu_1(a)$ and $\mathcal{M}\{a \in \xi_2\} = \mu_2(a)$ immediately.

Inference Rule 11.4 *Let $\mathbb{X}_1, \mathbb{X}_2, \cdots, \mathbb{X}_m$ be concepts. Assume rules "if $\mathbb{X}_1$ is ξ_{i1} and $\cdots$ and $\mathbb{X}_m$ is ξ_{im} then $\mathbb{Y}$ is η_i" for $i = 1, 2, \cdots, k$. From $\mathbb{X}_1$ is a_1 and $\cdots$ and $\mathbb{X}_m$ is a_m we infer that $\mathbb{Y}$ is an uncertain set*

$$\eta^* = \sum_{i=1}^{k} \frac{c_i \cdot \eta_i|_{(a_1\in\xi_{i1})\cap(a_2\in\xi_{i2})\cap\cdots\cap(a_m\in\xi_{im})}}{c_1 + c_2 + \cdots + c_k} \tag{11.13}$$

where the coefficients are determined by

$$c_i = \mathcal{M}\left\{(a_1 \in \xi_{i1}) \cap (a_2 \in \xi_{i2}) \cap \cdots \cap (a_m \in \xi_{im})\right\} \tag{11.14}$$

for $i = 1, 2, \cdots, k$. The inference rule is represented by

$$\begin{array}{l} \text{Rule 1: If } \mathbb{X}_1 \text{ is } \xi_{11} \text{ and } \cdots \text{ and } \mathbb{X}_m \text{ is } \xi_{1m} \text{ then } \mathbb{Y} \text{ is } \eta_1 \\ \text{Rule 2: If } \mathbb{X}_1 \text{ is } \xi_{21} \text{ and } \cdots \text{ and } \mathbb{X}_m \text{ is } \xi_{2m} \text{ then } \mathbb{Y} \text{ is } \eta_2 \\ \quad\cdots \\ \text{Rule } k\text{: If } \mathbb{X}_1 \text{ is } \xi_{k1} \text{ and } \cdots \text{ and } \mathbb{X}_m \text{ is } \xi_{km} \text{ then } \mathbb{Y} \text{ is } \eta_k \\ \text{From: } \mathbb{X}_1 \text{ is } a_1 \text{ and } \cdots \text{ and } \mathbb{X}_m \text{ is } a_m \\ \hline \text{Infer: } \mathbb{Y} \text{ is } \eta^* \text{ determined by (11.13)} \end{array} \tag{11.15}$$

Theorem 11.4 *Assume $\xi_{i1}, \xi_{i2}, \cdots, \xi_{im}, \eta_i$ are independent uncertain sets with membership functions $\mu_{i1}, \mu_{i2}, \cdots, \mu_{im}, \nu_i$, $i = 1, 2, \cdots, k$, respectively.*

If $\xi_1^*, \xi_2^*, \cdots, \xi_m^*$ are constants $a_1, a_2, \cdots, a_m$, respectively, then the inference rule 11.4 yields

$$\eta^* = \sum_{i=1}^{k} \frac{c_i \cdot \eta_i^*}{c_1 + c_2 + \cdots + c_k} \tag{11.16}$$

where η_i^* are uncertain sets whose membership functions are given by

$$\nu_i^*(y) = \begin{cases} \dfrac{\nu_i(y)}{c_i}, & \text{if } \nu_i(y) < c_i/2 \\ \dfrac{\nu_i(y) + c_i - 1}{c_i}, & \text{if } \nu_i(y) > 1 - c_i/2 \\ 0.5, & \text{otherwise} \end{cases} \tag{11.17}$$

and c_i are constants determined by

$$c_i = \min_{1 \le l \le m} \mu_{il}(a_l) \tag{11.18}$$

for $i = 1, 2, \cdots, k$, respectively.

Proof: For each i, since $a_1 \in \xi_{i1}, a_2 \in \xi_{i2}, \cdots, a_m \in \xi_{im}$ are independent events, we immediately have

$$\mathcal{M}\left\{\bigcap_{j=1}^{m} (a_j \in \xi_{ij})\right\} = \min_{1 \le j \le m} \mathcal{M}\{a_j \in \xi_{ij}\} = \min_{1 \le l \le m} \mu_{il}(a_l)$$

for $i = 1, 2, \cdots, k$. From those equations, we may prove the theorem by the inference rule 11.4 immediately.

11.2 Uncertain System

Uncertain system, proposed by Liu [127], is a function from its inputs to outputs based on the uncertain inference rule. Usually, an uncertain system consists of 5 parts:

1. inputs that are crisp data to be fed into the uncertain system;
2. a rule-base that contains a set of if-then rules provided by the experts;
3. an uncertain inference rule that infers uncertain consequents from the uncertain antecedents;
4. an expected value operator that converts the uncertain consequents to crisp values;
5. outputs that are crisp data yielded from the expected value operator.

Now let us consider an uncertain system in which there are m crisp inputs $\alpha_1, \alpha_2, \cdots, \alpha_m$, and n crisp outputs $\beta_1, \beta_2, \cdots, \beta_n$. At first, we infer n uncertain sets $\eta_1^*, \eta_2^*, \cdots, \eta_n^*$ from the m crisp inputs by the rule-base (i.e., a set of if-then rules),

$$\begin{array}{l}\text{If } \xi_{11} \text{ and } \xi_{12} \text{ and} \cdots \text{and } \xi_{1m} \text{ then } \eta_{11} \text{ and } \eta_{12} \text{ and} \cdots \text{and } \eta_{1n} \\ \text{If } \xi_{21} \text{ and } \xi_{22} \text{ and} \cdots \text{and } \xi_{2m} \text{ then } \eta_{21} \text{ and } \eta_{22} \text{ and} \cdots \text{and } \eta_{2n} \\ \quad \cdots \\ \text{If } \xi_{k1} \text{ and } \xi_{k2} \text{ and} \cdots \text{and } \xi_{km} \text{ then } \eta_{k1} \text{ and } \eta_{k2} \text{ and} \cdots \text{and } \eta_{kn}\end{array} \tag{11.19}$$

and the uncertain inference rule

$$\eta_j^* = \sum_{i=1}^{k} \frac{c_i \cdot \eta_{ij}|_{(\alpha_1 \in \xi_{i1}) \cap (\alpha_2 \in \xi_{i2}) \cap \cdots \cap (\alpha_m \in \xi_{im})}}{c_1 + c_2 + \cdots + c_k} \tag{11.20}$$

for $j = 1, 2, \cdots, n$, where the coefficients are determined by

$$c_i = \mathcal{M}\{(\alpha_1 \in \xi_{i1}) \cap (\alpha_2 \in \xi_{i2}) \cap \cdots \cap (\alpha_m \in \xi_{im})\} \tag{11.21}$$

for $i = 1, 2, \cdots, k$. Thus by using the expected value operator, we obtain

$$\beta_j = E[\eta_j^*] \tag{11.22}$$

for $j = 1, 2, \cdots, n$. Until now we have constructed a function from inputs $\alpha_1, \alpha_2, \cdots, \alpha_m$ to outputs $\beta_1, \beta_2, \cdots, \beta_n$. Write this function by f, i.e.,

$$(\beta_1, \beta_2, \cdots, \beta_n) = f(\alpha_1, \alpha_2, \cdots, \alpha_m). \tag{11.23}$$

Then we get an uncertain system f.

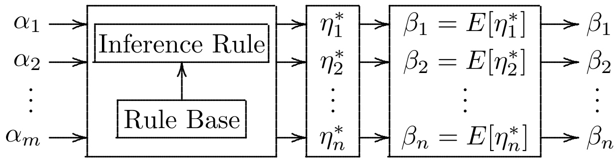


Figure 11.2: An Uncertain System. Reprinted from Liu [129].

Theorem 11.5 *Assume $\xi_{i1}, \xi_{i2}, \cdots, \xi_{im}, \eta_{i1}, \eta_{i2}, \cdots, \eta_{in}$ are independent uncertain sets with membership functions $\mu_{i1}, \mu_{i2}, \cdots, \mu_{im}, \nu_{i1}, \nu_{i2}, \cdots, \nu_{in}$, $i = 1, 2, \cdots, k$, respectively. Then the uncertain system from $(\alpha_1, \alpha_2, \cdots, \alpha_m)$ to $(\beta_1, \beta_2, \cdots, \beta_n)$ is*

$$\beta_j = \sum_{i=1}^{k} \frac{c_i \cdot E[\eta_{ij}^*]}{c_1 + c_2 + \cdots + c_k} \tag{11.24}$$

for $j = 1, 2, \cdots, n$*, where* η^*_{ij} *are uncertain sets whose membership functions are given by*

$$\nu^*_{ij}(y) = \begin{cases} \dfrac{\nu_{ij}(y)}{c_i}, & \text{if } \nu_{ij}(y) < c_i/2 \\ \dfrac{\nu_{ij}(y) + c_i - 1}{c_i}, & \text{if } \nu_{ij}(y) > 1 - c_i/2 \\ 0.5, & \text{otherwise} \end{cases} \tag{11.25}$$

and c_i *are constants determined by*

$$c_i = \min_{1 \le l \le m} \mu_{il}(\alpha_l) \tag{11.26}$$

for $i = 1, 2, \cdots, k$*,* $j = 1, 2, \cdots, n$*, respectively.*

Proof: It follows from the inference rule 11.4 that the uncertain sets η^*_j are

$$\eta^*_j = \sum_{i=1}^{k} \frac{c_i \cdot \eta^*_{ij}}{c_1 + c_2 + \cdots + c_k}$$

for $j = 1, 2, \cdots, n$. Since $\eta^*_{ij}, i = 1, 2, \cdots, k, j = 1, 2, \cdots, n$ are independent uncertain sets, we get the theorem immediately by the linearity of expected value operator.

Remark 11.1: The uncertain system allows the uncertain sets η_{ij} in the rule-base (11.19) become constants b_{ij}, i.e.,

$$\eta_{ij} = b_{ij} \tag{11.27}$$

for $i = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, n$. In this case, the uncertain system (11.24) becomes

$$\beta_j = \sum_{i=1}^{k} \frac{c_i \cdot b_{ij}}{c_1 + c_2 + \cdots + c_k} \tag{11.28}$$

for $j = 1, 2, \cdots, n$.

Remark 11.2: The uncertain system allows the uncertain sets η_{ij} in the rule-base (11.19) become functions h_{ij} of inputs $\alpha_1, \alpha_2, \cdots, \alpha_m$, i.e.,

$$\eta_{ij} = h_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_m) \tag{11.29}$$

for $i = 1, 2, \cdots, k$ and $j = 1, 2, \cdots, n$. In this case, the uncertain system (11.24) becomes

$$\beta_j = \sum_{i=1}^{k} \frac{c_i \cdot h_{ij}(\alpha_1, \alpha_2, \cdots, \alpha_m)}{c_1 + c_2 + \cdots + c_k} \tag{11.30}$$

for $j = 1, 2, \cdots, n$.

Uncertain Systems are Universal Approximator

Uncertain systems are capable of approximating any continuous function on a compact set (i.e., bounded and closed set) to arbitrary accuracy. This is the reason why uncertain systems may play a controller. The following theorem shows this fact.

Theorem 11.6 *(Peng and Chen [186]) For any given continuous function g on a compact set $D \subset \Re^m$ and any given $\varepsilon > 0$, there exists an uncertain system f such that*

$$\|f(\alpha_1, \alpha_2, \cdots, \alpha_m) - g(\alpha_1, \alpha_2, \cdots, \alpha_m)\| < \varepsilon \tag{11.31}$$

for any $(\alpha_1, \alpha_2, \cdots, \alpha_m) \in D$.

Proof: Without loss of generality, we assume that the function g is a real-valued function with only two variables α_1 and α_2, and the compact set is a unit rectangle $D = [0,1] \times [0,1]$. Since g is continuous on D and then is uniformly continuous, for any given number $\varepsilon > 0$, there is a number $\delta > 0$ such that

$$|g(\alpha_1, \alpha_2) - g(\alpha_1', \alpha_2')| < \varepsilon \tag{11.32}$$

whenever $\|(\alpha_1, \alpha_2) - (\alpha_1', \alpha_2')\| < \delta$. Let k be an integer larger than $1/(\sqrt{2}\delta)$, and write

$$D_{ij} = \left\{(\alpha_1, \alpha_2) \mid \frac{i-1}{k} < \alpha_1 \le \frac{i}{k}, \frac{j-1}{k} < \alpha_2 \le \frac{j}{k}\right\} \tag{11.33}$$

for $i, j = 1, 2, \cdots, k$. Note that $\{D_{ij}\}$ is a sequence of disjoint rectangles whose "diameter" is less than δ. Define uncertain sets

$$\xi_i = \left(\frac{i-1}{k}, \frac{i}{k}\right), \quad i = 1, 2, \cdots, k, \tag{11.34}$$

$$\eta_j = \left(\frac{j-1}{k}, \frac{j}{k}\right), \quad j = 1, 2, \cdots, k. \tag{11.35}$$

Then we assume a rule-base with $k \times k$ if-then rules,

$$\text{Rule } ij\text{: If } \xi_i \text{ and } \eta_j \text{ then } g(i/k, j/k), \quad i, j = 1, 2, \cdots, k. \tag{11.36}$$

According to the uncertain inference rule, the corresponding uncertain system from D to $\Re$ is

$$f(\alpha_1, \alpha_2) = g(i/k, j/k), \quad \text{if } (\alpha_1, \alpha_2) \in D_{ij},\ i, j = 1, 2, \cdots, k. \tag{11.37}$$

It follows from (11.32) that for any $(\alpha_1, \alpha_2) \in D_{ij} \subset D$, we have

$$|f(\alpha_1, \alpha_2) - g(\alpha_1, \alpha_2)| = |g(i/k, j/k) - g(\alpha_1, \alpha_2)| < \varepsilon. \tag{11.38}$$

The theorem is thus verified. Hence uncertain systems are universal approximators!

11.3 Uncertain Control

Uncertain controller, designed by Liu [127], is a special uncertain system that maps the state variables of a process under control to the action variables. Thus an uncertain controller consists of the same 5 parts of uncertain system: inputs, a rule-base, an uncertain inference rule, an expected value operator, and outputs. The distinguished point is that the inputs of uncertain controller are the state variables of the process under control, and the outputs are the action variables.

Figure 11.3 shows an uncertain control system consisting of an uncertain controller and a process. Note that t represents time, $\alpha_1(t), \alpha_2(t), \cdots, \alpha_m(t)$ are not only the inputs of uncertain controller but also the outputs of process, and $\beta_1(t), \beta_2(t), \cdots, \beta_n(t)$ are not only the outputs of uncertain controller but also the inputs of process.

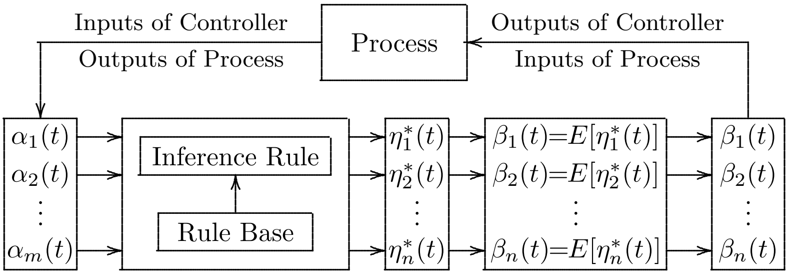


Figure 11.3: An Uncertain Control System. Reprinted from Liu [129].

11.4 Inverted Pendulum

Inverted pendulum system is a nonlinear unstable system that is widely used as a benchmark for testing control algorithms. Many good techniques already exist for balancing inverted pendulum. Among others, Gao [52] successfully balanced an inverted pendulum by the uncertain controller with 5×5 if-then rules.

The uncertain controller has two inputs ("angle" and "angular velocity") and one output ("force"). Three of them will be represented by uncertain sets labeled by

"negative large"	NL
"negative small"	NS
"zero"	Z
"positive small"	PS
"positive large"	PL

The membership functions of those uncertain sets are shown in Figures 11.5, 11.6 and 11.7.

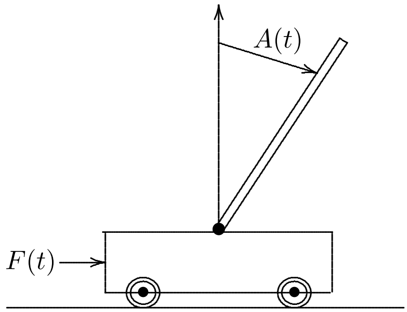


Figure 11.4: An Inverted Pendulum in which $A(t)$ represents the angular position and $F(t)$ represents the force that moves the cart at time t. Reprinted from Liu [129].

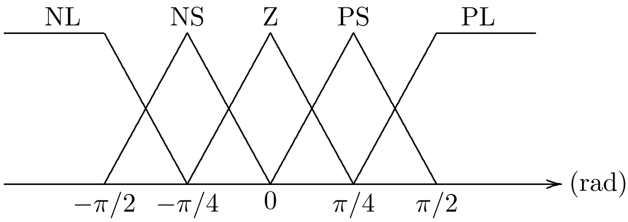


Figure 11.5: Membership Functions of "Angle"

Intuitively, when the inverted pendulum has a large clockwise angle and a large clockwise angular velocity, we should give it a large force to the right. Thus we have an if-then rule,

If the angle is negative large
and the angular velocity is negative large,
then the force is positive large.

Similarly, when the inverted pendulum has a large counterclockwise angle and a large counterclockwise angular velocity, we should give it a large force to the left. Thus we have an if-then rule,

If the angle is positive large
and the angular velocity is positive large,
then the force is negative large.

Note that each input or output has 5 states and each state is represented by an uncertain set. This implies that the rule-base contains 5×5 if-then rules. In order to balance the inverted pendulum, the 25 if-then rules in Table 11.1 are accepted.

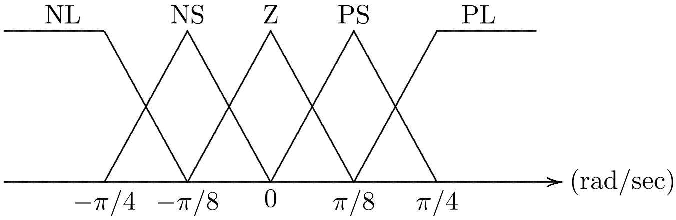


Figure 11.6: Membership Functions of "Angular Velocity"

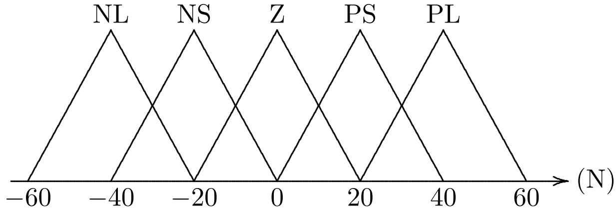


Figure 11.7: Membership Functions of "Force"

A lot of simulation results show that the uncertain controller may balance the inverted pendulum successfully.

11.5 Bibliographic Notes

The basic uncertain inference rule was initialized by Liu [127] in 2010 by the tool of conditional uncertain set. After that, Gao, Gao and Ralescu [49] extended the uncertain inference rule to the case with multiple antecedents and multiple if-then rules.

Based on the uncertain inference rules, Liu [127] suggested the concept of uncertain system, and then presented the tool of uncertain controller. As an important contribution, Peng and Chen [186] proved that uncertain systems

Table 11.1: Rule Base with 5×5 If-Then Rules

angle \ velocity	NL	NS	Z	PS	PL
NL	PL	PL	PL	PS	Z
NS	PL	PL	PS	Z	NS
Z	PL	PS	Z	NS	NL
PS	PS	Z	NS	NL	NL
PL	Z	NS	NL	NL	NL

are universal approximator and then demonstrated that the uncertain controller is a reasonable tool. As a successful application, Gao [52] balanced an inverted pendulum by using the uncertain controller.

Chapter 12

Uncertain Process

The study of uncertain process was started by Liu [123] in 2008 for modeling the evolution of uncertain phenomena. This chapter will give the concept of uncertain process, and introduce sample path, uncertainty distribution, independent increment, stationary increment, extreme value, first hitting time, and time integral of uncertain process.

12.1 Uncertain Process

An uncertain process is essentially a sequence of uncertain variables indexed by time. A formal definition is given below.

Definition 12.1 *(Liu [123]) Let* $(\Gamma, \mathcal{L}, \mathcal{M})$ *be an uncertainty space and let* T *be a totally ordered set (e.g. time). An uncertain process is a function* $X_t(\gamma)$ *from* $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ *to the set of real numbers such that* $\{X_t \in B\}$ *is an event for any Borel set* B *at each time* t*.*

Remark 12.1: If X_t is an uncertain process, then X_t is an uncertain variable at each time t.

Example 12.1: Let a and b be real numbers with $a < b$. Assume X_t is a linear uncertain variable, i.e.,

$$X_t \sim \mathcal{L}(at, bt) \tag{12.1}$$

at each time t. Then X_t is an uncertain process.

Example 12.2: Let a, b, c be real numbers with $a < b < c$. Assume X_t is a zigzag uncertain variable, i.e.,

$$X_t \sim \mathcal{Z}(at, bt, ct) \tag{12.2}$$

at each time t. Then X_t is an uncertain process.

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5_13

Example 12.3: Let e and σ be real numbers with $\sigma > 0$. Assume X_t is a normal uncertain variable, i.e.,

$$X_t \sim \mathcal{N}(et, \sigma t) \tag{12.3}$$

at each time t. Then X_t is an uncertain process.

Example 12.4: Let e and σ be real numbers with $\sigma > 0$. Assume X_t is a lognormal uncertain variable, i.e.,

$$X_t \sim \mathcal{LOGN}(et, \sigma t) \tag{12.4}$$

at each time t. Then X_t is an uncertain process.

Sample Path

Definition 12.2 *(Liu [123]) Let X_t be an uncertain process. Then for each $\gamma \in \Gamma$, the function $X_t(\gamma)$ is called a sample path of X_t.*

Note that each sample path is a real-valued function of time t. In addition, an uncertain process may also be regarded as a function from an uncertainty space to a collection of sample paths.

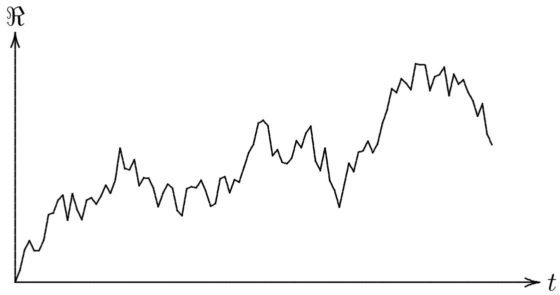


Figure 12.1: A Sample Path of Uncertain Process. Reprinted from Liu [129].

Definition 12.3 *An uncertain process X_t is said to be sample-continuous if almost all sample paths are continuous functions with respect to time t.*

Uncertain Field

Uncertain field is a generalization of uncertain process when the index set T becomes a partially ordered set (e.g. time × space, or a surface).

Definition 12.4 *(Liu [139]) Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and let T be a partially ordered set (e.g. time × space). An uncertain field is a function $X_t(\gamma)$ from $T \times (\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers such that $\{X_t \in B\}$ is an event for any Borel set B at each time t.*

12.2 Uncertainty Distribution

An uncertainty distribution of uncertain process is a sequence of uncertainty distributions of uncertain variables indexed by time. Thus an uncertainty distribution of uncertain process is a surface rather than a curve. A formal definition is given below.

Definition 12.5 *(Liu [139]) An uncertain process X_t is said to have an uncertainty distribution $\Phi_t(x)$ if at each time t, the uncertain variable X_t has the uncertainty distribution $\Phi_t(x)$.*

Example 12.5: The linear uncertain process $X_t \sim \mathcal{L}(at, bt)$ has an uncertainty distribution,

$$\Phi_t(x) = \begin{cases} 0, & \text{if } x \le at \\ \dfrac{x - at}{(b-a)t}, & \text{if } at \le x \le bt \\ 1, & \text{if } x \ge bt. \end{cases} \tag{12.5}$$

Example 12.6: The zigzag uncertain process $X_t \sim \mathcal{Z}(at, bt, ct)$ has an uncertainty distribution,

$$\Phi_t(x) = \begin{cases} 0, & \text{if } x \le at \\ \dfrac{x - at}{2(b-a)t}, & \text{if } at \le x \le bt \\ \dfrac{x + ct - 2bt}{2(c-b)t}, & \text{if } bt \le x \le ct \\ 1, & \text{if } x \ge ct. \end{cases} \tag{12.6}$$

Example 12.7: The normal uncertain process $X_t \sim \mathcal{N}(et, \sigma t)$ has an uncertainty distribution,

$$\Phi_t(x) = \left(1 + \exp\left(\frac{\pi(et - x)}{\sqrt{3}\sigma t}\right)\right)^{-1}. \tag{12.7}$$

Example 12.8: The lognormal uncertain process $X_t \sim \mathcal{LOGN}(et, \sigma t)$ has an uncertainty distribution,

$$\Phi_t(x) = \left(1 + \exp\left(\frac{\pi(et - \ln x)}{\sqrt{3}\sigma t}\right)\right)^{-1}. \tag{12.8}$$

Theorem 12.1 *(Liu [139], Sufficient and Necessary Condition) A function* $\Phi_t(x) : T \times \Re \to [0,1]$ *is an uncertainty distribution of uncertain process if and only if at each time* t*, it is a monotone increasing function with respect to* x *except* $\Phi_t(x) \equiv 0$ *and* $\Phi_t(x) \equiv 1$*.*

Proof: If $\Phi_t(x)$ is an uncertainty distribution of some uncertain process X_t, then at each time t, $\Phi_t(x)$ is the uncertainty distribution of uncertain variable X_t. It follows from Peng-Iwamura theorem that $\Phi_t(x)$ is a monotone increasing function with respect to x and $\Phi_t(x) \not\equiv 0$, $\Phi_t(x) \not\equiv 1$. Conversely, if at each time t, $\Phi_t(x)$ is a monotone increasing function except $\Phi_t(x) \equiv 0$ and $\Phi_t(x) \equiv 1$, it follows from Peng-Iwamura theorem that there exists an uncertain variable ξ_t whose uncertainty distribution is just $\Phi_t(x)$. Define

$$X_t = \xi_t, \quad \forall t \in T.$$

Then X_t is an uncertain process and has the uncertainty distribution $\Phi_t(x)$. The theorem is verified.

Theorem 12.2 *Let* X_t *be an uncertain process with uncertainty distribution* $\Phi_t(x)$*, and let* $f(x)$ *be a measurable function. Then* $f(X_t)$ *is also an uncertain process. Furthermore, (i) if* $f(x)$ *is a strictly increasing function, then* $f(X_t)$ *has an uncertainty distribution*

$$\Psi_t(x) = \Phi_t(f^{-1}(x)); \tag{12.9}$$

and (ii) if $f(x)$ *is a strictly decreasing function and* $\Phi_t(x)$ *is continuous with respect to* x*, then* $f(X_t)$ *has an uncertainty distribution*

$$\Psi_t(x) = 1 - \Phi_t(f^{-1}(x)). \tag{12.10}$$

Proof: At each time t, since X_t is an uncertain variable, it follows from Theorem 2.1 that $f(X_t)$ is also an uncertain variable. Thus $f(X_t)$ is an uncertain process. The equations (12.9) and (12.10) may be verified by the operational law of uncertain variables immediately.

Example 12.9: Let X_t be an uncertain process with uncertainty distribution $\Phi_t(x)$. Show that the uncertain process $aX_t + b$ has an uncertainty distribution,

$$\Psi_t(x) = \begin{cases} \Phi_t((x-b)/a), & \text{if } a > 0 \\ 1 - \Phi_t((x-b)/a), & \text{if } a < 0. \end{cases} \tag{12.11}$$

Regular Uncertainty Distribution

Definition 12.6 *(Liu [139]) An uncertainty distribution* $\Phi_t(x)$ *is said to be regular if at each time* t*, it is a continuous and strictly increasing function with respect to* x *at which* $0 < \Phi_t(x) < 1$*, and*

$$\lim_{x\to-\infty} \Phi_t(x) = 0, \quad \lim_{x\to+\infty} \Phi_t(x) = 1. \tag{12.12}$$

It is clear that linear uncertainty distribution, zigzag uncertainty distribution, normal uncertainty distribution and lognormal uncertainty distribution of uncertain process are all regular.

Note that we have stipulated that a crisp initial value X_0 has a regular uncertainty distribution. That is, we allow the initial value of regular uncertain process to be a constant whose uncertainty distribution is

$$\Phi_0(x) = \begin{cases} 1, & \text{if } x \geq X_0 \\ 0, & \text{if } x < X_0 \end{cases} \tag{12.13}$$

and say $\Phi_0(x)$ is a continuous and strictly increasing function with respect to x at which $0 < \Phi_0(x) < 1$ even though it is discontinuous at X_0.

Inverse Uncertainty Distribution

Definition 12.7 *(Liu [139]) Let X_t be an uncertain process with regular uncertainty distribution $\Phi_t(x)$. Then the inverse function $\Phi_t^{-1}(\alpha)$ is called the inverse uncertainty distribution of X_t.*

Note that at each time t, the inverse uncertainty distribution $\Phi_t^{-1}(\alpha)$ is well defined on the open interval $(0, 1)$. If needed, we may extend the domain to $[0, 1]$ via

$$\Phi_t^{-1}(0) = \lim_{\alpha \downarrow 0} \Phi_t^{-1}(\alpha), \quad \Phi_t^{-1}(1) = \lim_{\alpha \uparrow 1} \Phi_t^{-1}(\alpha). \tag{12.14}$$

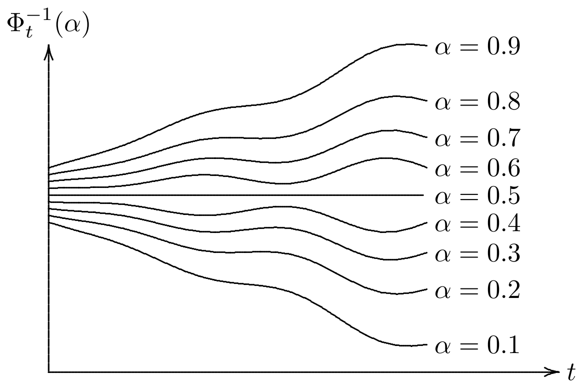


Figure 12.2: Inverse Uncertainty Distribution of Uncertain Process

Example 12.10: The linear uncertain process $X_t \sim \mathcal{L}(at, bt)$ has an inverse uncertainty distribution,

$$\Phi_t^{-1}(\alpha) = (1 - \alpha)at + \alpha bt. \tag{12.15}$$

Example 12.11: The zigzag uncertain process $X_t \sim \mathcal{Z}(at, bt, ct)$ has an inverse uncertainty distribution,

$$\Phi_t^{-1}(\alpha) = \begin{cases} (1-2\alpha)at + 2\alpha bt, & \text{if } \alpha < 0.5 \\ (2-2\alpha)bt + (2\alpha-1)ct, & \text{if } \alpha \ge 0.5. \end{cases} \tag{12.16}$$

Example 12.12: The normal uncertain process $X_t \sim \mathcal{N}(et, \sigma t)$ has an inverse uncertainty distribution,

$$\Phi_t^{-1}(\alpha) = et + \frac{\sigma t\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}. \tag{12.17}$$

Example 12.13: The lognormal uncertain process $X_t \sim \mathcal{LOGN}(et, \sigma t)$ has an inverse uncertainty distribution,

$$\Phi_t^{-1}(\alpha) = \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right). \tag{12.18}$$

Theorem 12.3 *(Liu [139], Sufficient and Necessary Condition) A function $\Phi_t^{-1}(\alpha) : T \times (0,1) \to \Re$ is an inverse uncertainty distribution of uncertain process if and only if at each time t, it is a continuous and strictly increasing function with respect to α.*

Proof: Suppose $\Phi_t^{-1}(\alpha)$ is an inverse uncertainty distribution of uncertain process X_t. Then at each time t, $\Phi_t^{-1}(\alpha)$ is an inverse uncertainty distribution of uncertain variable X_t. It follows from Theorem 2.6 that $\Phi_t^{-1}(\alpha)$ is a continuous and strictly increasing function with respect to $\alpha \in (0,1)$. Conversely, if $\Phi_t^{-1}(\alpha)$ is a continuous and strictly increasing function with respect to $\alpha \in (0,1)$, it follows from Theorem 2.6 that there exists an uncertain variable ξ_t whose inverse uncertainty distribution is just $\Phi_t^{-1}(\alpha)$. Define

$$X_t = \xi_t, \quad \forall t \in T.$$

Then X_t is an uncertain process and has the inverse uncertainty distribution $\Phi_t^{-1}(\alpha)$. The theorem is proved.

Remark 12.2: Note that we stipulate that a crisp initial value X_0 has an inverse uncertainty distribution

$$\Phi_0^{-1}(\alpha) \equiv X_0 \tag{12.19}$$

and say $\Phi_0^{-1}(\alpha)$ is a continuous and strictly increasing function with respect to $\alpha \in (0,1)$ even though it is not.

12.3 Independence and Operational Law

Definition 12.8 *(Liu [139]) Uncertain processes $X_{1t}, X_{2t}, \cdots, X_{nt}$ are said to be independent if for any positive integer k and any times $t_1, t_2, \cdots, t_k$, the uncertain vectors*

$$\boldsymbol{\xi}_i = (X_{it_1}, X_{it_2}, \cdots, X_{it_k}), \quad i = 1, 2, \cdots, n \tag{12.20}$$

are independent, i.e., for any k-dimensional Borel sets $B_1, B_2, \cdots, B_n$, we have

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in B_i\}. \tag{12.21}$$

Exercise 12.1: Let $X_{1t}, X_{2t}, \cdots, X_{nt}$ be independent uncertain processes, and let $t_1, t_2, \cdots, t_n$ be any times. Show that

$$X_{1t_1}, X_{2t_2}, \cdots, X_{nt_n} \tag{12.22}$$

are independent uncertain variables.

Exercise 12.2: Let X_t and Y_t be independent uncertain processes. For any times $t_1, t_2, \cdots, t_k$ and $s_1, s_2, \cdots, s_m$, show that

$$(X_{t_1}, X_{t_2}, \cdots, X_{t_k}) \text{ and } (Y_{s_1}, Y_{s_2}, \cdots, Y_{s_m}) \tag{12.23}$$

are independent uncertain vectors.

Theorem 12.4 *(Liu [139]) Uncertain processes $X_{1t}, X_{2t}, \cdots, X_{nt}$ are independent if and only if for any positive integer k, any times $t_1, t_2, \cdots, t_k$, and any k-dimensional Borel sets $B_1, B_2, \cdots, B_n$, we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\boldsymbol{\xi}_i \in B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\boldsymbol{\xi}_i \in B_i\} \tag{12.24}$$

where $\boldsymbol{\xi}_i = (X_{it_1}, X_{it_2}, \cdots, X_{it_k})$ for $i = 1, 2, \cdots, n$.

Proof: It follows from Theorem 2.64 that $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_n$ are independent uncertain vectors if and only if (12.24) holds. The theorem is thus verified.

Theorem 12.5 *(Liu [139], Operational Law) Let $X_{1t}, X_{2t}, \cdots, X_{nt}$ be independent uncertain processes with regular uncertainty distributions $\Phi_{1t}, \Phi_{2t}, \cdots, \Phi_{nt}$, respectively. If the function $f(x_1, x_2, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$, then the uncertain process*

$$X_t = f(X_{1t}, X_{2t}, \cdots, X_{nt}) \tag{12.25}$$

has an inverse uncertainty distribution

$$\Phi_t^{-1}(\alpha) = f(\Phi_{1t}^{-1}(\alpha), \cdots, \Phi_{mt}^{-1}(\alpha), \Phi_{m+1,t}^{-1}(1-\alpha), \cdots, \Phi_{nt}^{-1}(1-\alpha)). \tag{12.26}$$

Proof: At any time t, it is clear that $X_{1t}, X_{2t}, \cdots, X_{nt}$ are independent uncertain variables with inverse uncertainty distributions $\Phi_{1t}^{-1}(\alpha), \Phi_{2t}^{-1}(\alpha), \cdots,$ $\Phi_{nt}^{-1}(\alpha)$, respectively. The theorem follows from the operational law of uncertain variables immediately.

12.4 Independent Increment Process

An independent increment process is an uncertain process that has independent increments. A formal definition is given below.

Definition 12.9 *(Liu [123]) An uncertain process X_t is said to have independent increments if*

$$X_{t_0},\ X_{t_1} - X_{t_0},\ X_{t_2} - X_{t_1},\ \cdots,\ X_{t_k} - X_{t_{k-1}} \tag{12.27}$$

are independent uncertain variables where t_0 is the initial time and $t_1, t_2, \cdots, t_k$ are any times with $t_0 < t_1 < \cdots < t_k$.

That is, an independent increment process means that its increments are independent uncertain variables whenever the time intervals do not overlap. Please note that the increments are also independent of the initial state.

Theorem 12.6 *Let X_t be an independent increment process. Then for any real numbers a and b, the uncertain process*

$$Y_t = aX_t + b \tag{12.28}$$

is also an independent increment process.

Proof: Since X_t is an independent increment process, the uncertain variables

$$X_{t_0},\ X_{t_1} - X_{t_0},\ X_{t_2} - X_{t_1},\ \cdots,\ X_{t_k} - X_{t_{k-1}}$$

are independent. It follows from $Y_t = aX_t + b$ and Theorem 2.8 that

$$Y_{t_0},\ Y_{t_1} - Y_{t_0},\ Y_{t_2} - Y_{t_1},\ \cdots,\ Y_{t_k} - Y_{t_{k-1}}$$

are also independent. That is, Y_t is an independent increment process.

Remark 12.3: Generally speaking, a nonlinear function of independent increment process does not necessarily have independent increments. A typical example is the square of independent increment process.

Theorem 12.7 *(Liu [139]) Let X_t be an independent increment process. Then for any times $s < t$, the uncertain variables X_s and $X_t - X_s$ are independent.*

Proof: Since X_t is an independent increment process, the initial value and increments

$$X_0,\, X_s - X_0,\, X_t - X_s$$

are independent. It follows from $X_s = X_0 + (X_s - X_0)$ that X_s and $X_t - X_s$ are independent uncertain variables.

Theorem 12.8 *(Liu [139], Sufficient and Necessary Condition) A function* $\Phi_t^{-1}(\alpha) : T \times (0,1) \to \Re$ *is an inverse uncertainty distribution of independent increment process if and only if (i) at each time* t, $\Phi_t^{-1}(\alpha)$ *is a continuous and strictly increasing function; and (ii) for any times* $s < t$, $\Phi_t^{-1}(\alpha) - \Phi_s^{-1}(\alpha)$ *is a monotone increasing function with respect to* α.

Proof: Let X_t be an independent increment process with inverse uncertainty distribution $\Phi_t^{-1}(\alpha)$. First, it follows from Theorem 12.3 that $\Phi_t^{-1}(\alpha)$ is a continuous and strictly increasing function with respect to α. Next, it follows from Theorem 12.7 that X_s and $X_t - X_s$ are independent uncertain variables. Since X_s has an inverse uncertainty distribution $\Phi_s^{-1}(\alpha)$ and $X_t = X_s + (X_t - X_s)$, for any $\alpha < \beta$, we immediately have

$$\Phi_t^{-1}(\beta) - \Phi_t^{-1}(\alpha) \ge \Phi_s^{-1}(\beta) - \Phi_s^{-1}(\alpha).$$

That is,

$$\Phi_t^{-1}(\beta) - \Phi_s^{-1}(\beta) \ge \Phi_t^{-1}(\alpha) - \Phi_s^{-1}(\alpha).$$

Hence $\Phi_t^{-1}(\alpha) - \Phi_s^{-1}(\alpha)$ is a monotone (not strictly) increasing function with respect to α.

Conversely, let us prove that there exists an independent increment process whose inverse uncertainty distribution is just $\Phi_t^{-1}(\alpha)$. Without loss of generality, we only consider the range of $t \in [0,1]$. Let n be a positive integer. Since $\Phi_t^{-1}(\alpha)$ is a continuous and strictly increasing function and $\Phi_t^{-1}(\alpha) - \Phi_s^{-1}(\alpha)$ is a monotone increasing function with respect to α, there exist independent uncertain variables $\xi_{0n}, \xi_{1n}, \cdots, \xi_{nn}$ such that ξ_{0n} has an inverse uncertainty distribution

$$\Upsilon_{0n}^{-1}(\alpha) = \Phi_0^{-1}(\alpha)$$

and ξ_{in} have uncertainty distributions

$$\Upsilon_{in}(x) = \sup\left\{\alpha \,|\, \Phi_{i/n}^{-1}(\alpha) - \Phi_{(i-1)/n}^{-1}(\alpha) = x\right\},$$

$i = 1, 2, \cdots, n$, respectively. Define an uncertain process

$$X_t^n = \begin{cases} \displaystyle\sum_{i=0}^{k} \xi_{in}, & \text{if } t = \dfrac{k}{n} \quad (k = 0, 1, \cdots, n) \\ \text{linear}, & \text{otherwise.} \end{cases}$$

It may prove that X_t^n converges in distribution as $n \to \infty$. Furthermore, we may verify that the limit is indeed an independent increment process and has the inverse uncertainty distribution $\Phi_t^{-1}(\alpha)$. The theorem is verified.

Remark 12.4: It follows from Theorem 12.8 that the uncertainty distribution of independent increment process has a horn-like shape, i.e., for any $s < t$ and $\alpha < \beta$, we have

$$\Phi_s^{-1}(\beta) - \Phi_s^{-1}(\alpha) < \Phi_t^{-1}(\beta) - \Phi_t^{-1}(\alpha). \tag{12.29}$$

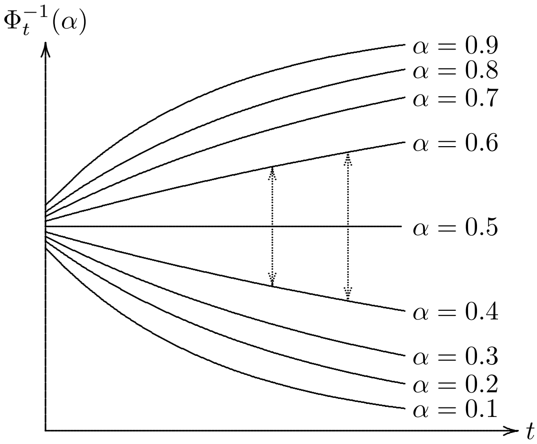


Figure 12.3: Inverse Uncertainty Distribution of Independent Increment Process: A Horn-like Family of Functions of t indexed by α

Exercise 12.3: Show that there exists an independent increment process with linear uncertainty distribution.

Exercise 12.4: Show that there exists an independent increment process with zigzag uncertainty distribution.

Exercise 12.5: Show that there exists an independent increment process with normal uncertainty distribution.

Exercise 12.6: Show that there does not exist an independent increment process with lognormal uncertainty distribution.

12.5 Stationary Independent Increment Process

An uncertain process X_t is said to have *stationary increments* if its increments are identically distributed uncertain variables whenever the time intervals have the same length, i.e., for any given $t > 0$, the increments $X_{s+t} - X_s$ are identically distributed uncertain variables for all $s > 0$.

Definition 12.10 *(Liu [123]) An uncertain process is said to be a stationary independent increment process if it has not only stationary increments but also independent increments.*

It is clear that a stationary independent increment process is a special independent increment process.

Theorem 12.9 *Let X_t be a stationary independent increment process. Then for any real numbers a and b, the uncertain process*

$$Y_t = aX_t + b \tag{12.30}$$

is also a stationary independent increment process.

Proof: Since X_t is an independent increment process, it follows from Theorem 12.6 that Y_t is also an independent increment process. On the other hand, since X_t is a stationary increment process, the increments $X_{s+t} - X_s$ are identically distributed uncertain variables for all $s > 0$. Thus

$$Y_{s+t} - Y_s = a(X_{s+t} - X_s)$$

are also identically distributed uncertain variables for all $s > 0$, and Y_t is a stationary increment process. Hence Y_t is a stationary independent increment process.

Theorem 12.10 *(Chen [17]) Suppose X_t is a stationary independent increment process. Then X_t and $(1-t)X_0 + tX_1$ are identically distributed uncertain variables for any time $t \geq 0$.*

Proof: We first prove the theorem when t is a rational number. Assume $t = q/p$ where p and q are irreducible integers. Let Φ be the common uncertainty distribution of increments

$$X_{1/p} - X_{0/p},\ X_{2/p} - X_{1/p},\ X_{3/p} - X_{2/p},\ \cdots$$

Then

$$X_t - X_0 = (X_{1/p} - X_{0/p}) + (X_{2/p} - X_{1/p}) + \cdots + (X_{q/p} - X_{(q-1)/p})$$

has an uncertainty distribution

$$\Psi(x) = \Phi(x/q). \tag{12.31}$$

In addition,

$$t(X_1 - X_0) = t((X_{1/p} - X_{0/p}) + (X_{2/p} - X_{1/p}) + \cdots + (X_{p/p} - X_{(p-1)/p}))$$

has an uncertainty distribution

$$\Upsilon(x) = \Phi(x/p/t) = \Phi(x/p/(q/p)) = \Phi(x/q). \tag{12.32}$$

It follows from (12.31) and (12.32) that $X_t - X_0$ and $t(X_1 - X_0)$ are identically distributed, and so are X_t and $(1-t)X_0 + tX_1$.

Remark 12.5: If X_t is a stationary independent increment process with $X_0 = 0$, then X_t/t and X_1 are identically distributed uncertain variables. In other words, there is an uncertainty distribution Φ such that

$$\frac{X_t}{t} \sim \Phi(x) \tag{12.33}$$

or equivalently,

$$X_t \sim \Phi\left(\frac{x}{t}\right) \tag{12.34}$$

for any time $t > 0$. Note that Φ is just the uncertainty distribution of X_1.

Theorem 12.11 *(Liu [139]) Let X_t be a stationary independent increment process whose initial value and increments have inverse uncertainty distributions. Then there exist two continuous and strictly increasing functions $\mu(\alpha)$ and $\nu(\alpha)$ such that X_t has an inverse uncertainty distribution*

$$\Phi_t^{-1}(\alpha) = \mu(\alpha) + \nu(\alpha)t. \tag{12.35}$$

Conversely, if there exist two continuous and strictly increasing functions $\mu(\alpha)$ and $\nu(\alpha)$ such that (12.35) holds, then there exists a stationary independent increment process X_t whose inverse uncertainty distribution is just $\Phi_t^{-1}(\alpha)$. Furthermore, X_t has a Lipschitz continuous version.

Proof: Assume X_t is a stationary independent increment process whose initial value and increments have inverse uncertainty distributions. Then X_0 and $X_1 - X_0$ are independent uncertain variables whose inverse uncertainty distributions exist and are denoted by $\mu(\alpha)$ and $\nu(\alpha)$, respectively. Then $\mu(\alpha)$ and $\nu(\alpha)$ are continuous and strictly increasing functions. Furthermore, it follows from Theorem 12.10 that X_t and $X_0 + (X_1 - X_0)t$ are identically distributed uncertain variables. Hence X_t has the inverse uncertainty distribution $\Phi_t^{-1}(\alpha) = \mu(\alpha) + \nu(\alpha)t$.

Conversely, let us prove that there exists a stationary independent increment process whose inverse uncertainty distribution is just $\Phi_t^{-1}(\alpha)$. Without loss of generality, we only consider the range of $t \in [0, 1]$. Let

$$\left\{\xi(r) \,\middle|\, r \text{ represents rational numbers in } [0,1]\right\}$$

be a countable sequence of independent uncertain variables, where $\xi(0)$ has an inverse uncertainty distribution $\mu(\alpha)$ and $\xi(r)$ have a common inverse uncertainty distribution $\nu(\alpha)$ for all rational numbers r in $(0, 1]$. For each positive integer n, we define an uncertain process

$$X_t^n = \begin{cases} \xi(0) + \dfrac{1}{n}\displaystyle\sum_{i=1}^{k} \xi\left(\dfrac{i}{n}\right), & \text{if } t = \dfrac{k}{n} \quad (k = 1, 2, \cdots, n) \\ \text{linear}, & \text{otherwise.} \end{cases}$$

It may prove that X_t^n converges in distribution as $n \to \infty$. Furthermore, we may verify that the limit is a stationary independent increment process and has the inverse uncertainty distribution $\Phi_t^{-1}(\alpha)$. The theorem is verified.

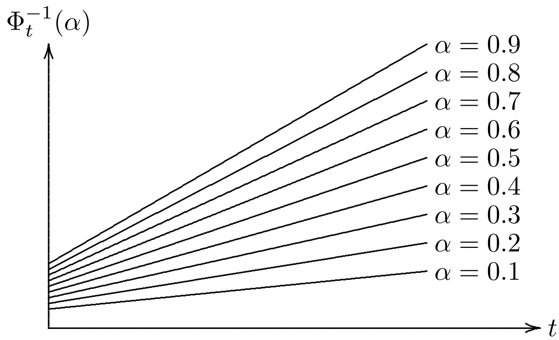


Figure 12.4: Inverse Uncertainty Distribution of Stationary Independent Increment Process: A Family of Linear Functions of t indexed by α

Exercise 12.7: Show that there exists a stationary independent increment process with linear uncertainty distribution.

Exercise 12.8: Show that there exists a stationary independent increment process with zigzag uncertainty distribution.

Exercise 12.9: Show that there exists a stationary independent increment process with normal uncertainty distribution.

Exercise 12.10: Show that there does not exist a stationary independent increment process with lognormal uncertainty distribution.

Theorem 12.12 *(Liu [129]) Let X_t be a stationary independent increment process. Then there exist two real numbers a and b such that*

$$E[X_t] = a + bt \tag{12.36}$$

for any time $t \geq 0$.

Proof: It follows from Theorem 12.10 that X_t and $X_0 + (X_1 - X_0)t$ are identically distributed uncertain variables. Thus we have

$$E[X_t] = E[X_0 + (X_1 - X_0)t].$$

Since X_0 and $X_1 - X_0$ are independent uncertain variables, we obtain

$$E[X_t] = E[X_0] + E[X_1 - X_0]t.$$

Hence (12.36) holds for $a = E[X_0]$ and $b = E[X_1 - X_0]$.

Theorem 12.13 *(Liu [129]) Let X_t be a stationary independent increment process with an initial value 0. Then for any times s and t, we have*

$$E[X_{s+t}] = E[X_s] + E[X_t]. \tag{12.37}$$

Proof: It follows from Theorem 12.12 that there exists a real number b such that $E[X_t] = bt$ for any time $t \geq 0$. Hence

$$E[X_{s+t}] = b(s + t) = bs + bt = E[X_s] + E[X_t].$$

Theorem 12.14 *(Chen [17]) Let X_t be a stationary independent increment process with a crisp initial value X_0. Then there exists a real number b such that*

$$V[X_t] = bt^2 \tag{12.38}$$

for any time $t \geq 0$.

Proof: It follows from Theorem 12.10 that X_t and $(1 - t)X_0 + tX_1$ are identically distributed uncertain variables. Since X_0 is a constant, we have

$$V[X_t] = V[(1 - t)X_0 + tX_1] = t^2V[X_1].$$

Hence (12.38) holds for $b = V[X_1]$.

Theorem 12.15 *(Chen [17]) Let X_t be a stationary independent increment process with a crisp initial value X_0. Then for any times s and t, we have*

$$\sqrt{V[X_{s+t}]} = \sqrt{V[X_s]} + \sqrt{V[X_t]}. \tag{12.39}$$

Proof: It follows from Theorem 12.14 that there exists a real number b such that $V[X_t] = bt^2$ for any time $t \geq 0$. Hence

$$\sqrt{V[X_{s+t}]} = \sqrt{b}(s + t) = \sqrt{b}s + \sqrt{b}t = \sqrt{V[X_s]} + \sqrt{V[X_t]}.$$

12.6 Extreme Value Theorem

This section will present a series of extreme value theorems for sample-continuous independent increment processes.

Theorem 12.16 *(Liu [135], Extreme Value Theorem) Let X_t be a sample-continuous independent increment process with uncertainty distribution $\Phi_t(x)$. Then the supremum*

$$\sup_{0\le t\le s} X_t \tag{12.40}$$

has an uncertainty distribution

$$\Psi(x) = \inf_{0\le t\le s} \Phi_t(x); \tag{12.41}$$

and the infimum

$$\inf_{0\le t\le s} X_t \tag{12.42}$$

has an uncertainty distribution

$$\Psi(x) = \sup_{0\le t\le s} \Phi_t(x). \tag{12.43}$$

Proof: Let $0 = t_1 < t_2 < \cdots < t_n = s$ be a partition of the closed interval $[0, s]$. It is clear that

$$X_{t_i} = X_{t_1} + (X_{t_2} - X_{t_1}) + \cdots + (X_{t_i} - X_{t_{i-1}})$$

for $i = 1, 2, \cdots, n$. Since the increments

$$X_{t_1},\ X_{t_2} - X_{t_1},\ \cdots, X_{t_n} - X_{t_{n-1}}$$

are independent uncertain variables, it follows from Theorem 2.15 that the maximum

$$\max_{1\le i\le n} X_{t_i}$$

has an uncertainty distribution

$$\min_{1\le i\le n} \Phi_{t_i}(x).$$

Since X_t is sample-continuous, we have

$$\max_{1\le i\le n} X_{t_i} \to \sup_{0\le t\le s} X_t$$

and

$$\min_{1\le i\le n} \Phi_{t_i}(x) \to \inf_{0\le t\le s} \Phi_t(x)$$

as $n \to \infty$. Thus (12.41) is proved. Similarly, it follows from Theorem 2.15 that the minimum

$$\min_{1\le i\le n} X_{t_i}$$

has an uncertainty distribution

$$\max_{1\le i\le n} \Phi_{t_i}(x).$$

Since X_t is sample-continuous, we have

$$\min_{1\le i\le n} X_{t_i} \to \inf_{0\le t\le s} X_t$$

and

$$\max_{1\le i\le n} \Phi_{t_i}(x) \to \sup_{0\le t\le s} \Phi_t(x)$$

as $n \to \infty$. Thus (12.43) is verified.

Theorem 12.17 *(Liu [135]) Let X_t be a sample-continuous independent increment process with uncertainty distribution $\Phi_t(x)$. If f is a strictly increasing function, then the supremum*

$$\sup_{0\le t\le s} f(X_t) \tag{12.44}$$

has an uncertainty distribution

$$\Psi(x) = \inf_{0\le t\le s} \Phi_t(f^{-1}(x)); \tag{12.45}$$

and the infimum

$$\inf_{0\le t\le s} f(X_t) \tag{12.46}$$

has an uncertainty distribution

$$\Psi(x) = \sup_{0\le t\le s} \Phi_t(f^{-1}(x)). \tag{12.47}$$

Proof: Since f is a strictly increasing function, $f(X_t) \le x$ if and only if $X_t \le f^{-1}(x)$. It follows from the extreme value theorem that

$$\begin{aligned}
\Psi(x) &= \mathcal{M}\left\{\sup_{0\le t\le s} f(X_t) \le x\right\} \\
&= \mathcal{M}\left\{\sup_{0\le t\le s} X_t \le f^{-1}(x)\right\} \\
&= \inf_{0\le t\le s} \Phi_t(f^{-1}(x)).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}\Psi(x) &= \mathcal{M}\left\{\inf_{0\le t\le s} f(X_t) \le x\right\}\\ &= \mathcal{M}\left\{\inf_{0\le t\le s} X_t \le f^{-1}(x)\right\}\\ &= \sup_{0\le t\le s} \Phi_t(f^{-1}(x)).\end{aligned}$$

The theorem is proved.

Exercise 12.11: Let X_t be a sample-continuous independent increment process with uncertainty distribution $\Phi_t(x)$. Show that the supremum

$$\sup_{0\le t\le s} \exp(X_t) \tag{12.48}$$

has an uncertainty distribution

$$\Psi(x) = \inf_{0\le t\le s} \Phi_t(\ln x); \tag{12.49}$$

and the infimum

$$\inf_{0\le t\le s} \exp(X_t) \tag{12.50}$$

has an uncertainty distribution

$$\Psi(x) = \sup_{0\le t\le s} \Phi_t(\ln x). \tag{12.51}$$

Exercise 12.12: Let X_t be a sample-continuous and positive independent increment process with uncertainty distribution $\Phi_t(x)$. Show that the supremum

$$\sup_{0\le t\le s} \ln X_t \tag{12.52}$$

has an uncertainty distribution

$$\Psi(x) = \inf_{0\le t\le s} \Phi_t(\exp(x)); \tag{12.53}$$

and the infimum

$$\inf_{0\le t\le s} \ln X_t \tag{12.54}$$

has an uncertainty distribution

$$\Psi(x) = \sup_{0\le t\le s} \Phi_t(\exp(x)). \tag{12.55}$$

Exercise 12.13: Let X_t be a sample-continuous and nonnegative independent increment process with uncertainty distribution $\Phi_t(x)$. Show that the supremum

$$\sup_{0\le t\le s} X_t^2 \tag{12.56}$$

has an uncertainty distribution

$$\Psi(x) = \inf_{0\le t\le s} \Phi_t(\sqrt{x}); \tag{12.57}$$

and the infimum

$$\inf_{0\le t\le s} X_t^2 \tag{12.58}$$

has an uncertainty distribution

$$\Psi(x) = \sup_{0\le t\le s} \Phi_t(\sqrt{x}). \tag{12.59}$$

Theorem 12.18 *(Liu [135]) Let X_t be a sample-continuous independent increment process with continuous uncertainty distribution $\Phi_t(x)$. If f is a strictly decreasing function, then the supremum*

$$\sup_{0\le t\le s} f(X_t) \tag{12.60}$$

has an uncertainty distribution

$$\Psi(x) = 1 - \sup_{0\le t\le s} \Phi_t(f^{-1}(x)); \tag{12.61}$$

and the infimum

$$\inf_{0\le t\le s} f(X_t) \tag{12.62}$$

has an uncertainty distribution

$$\Psi(x) = 1 - \inf_{0\le t\le s} \Phi_t(f^{-1}(x)). \tag{12.63}$$

Proof: Since f is a strictly decreasing function, $f(X_t) \le x$ if and only if $X_t \ge f^{-1}(x)$. It follows from the extreme value theorem that

$$\begin{aligned}\Psi(x) &= \mathcal{M}\left\{\sup_{0\le t\le s} f(X_t) \le x\right\} = \mathcal{M}\left\{\inf_{0\le t\le s} X_t \ge f^{-1}(x)\right\}\\ &= 1 - \mathcal{M}\left\{\inf_{0\le t\le s} X_t < f^{-1}(x)\right\} = 1 - \sup_{0\le t\le s} \Phi_t(f^{-1}(x)).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\Psi(x) &= \mathcal{M}\left\{\inf_{0\le t\le s} f(X_t) \le x\right\} = \mathcal{M}\left\{\sup_{0\le t\le s} X_t \ge f^{-1}(x)\right\}\\ &= 1 - \mathcal{M}\left\{\sup_{0\le t\le s} X_t < f^{-1}(x)\right\} = 1 - \inf_{0\le t\le s} \Phi_t(f^{-1}(x)).\end{aligned}$$

The theorem is proved.

Exercise 12.14: Let X_t be a sample-continuous independent increment process with continuous uncertainty distribution $\Phi_t(x)$. Show that the supremum

$$\sup_{0\le t\le s}\exp(-X_t) \tag{12.64}$$

has an uncertainty distribution

$$\Psi(x) = 1 - \sup_{0\le t\le s}\Phi_t(-\ln x); \tag{12.65}$$

and the infimum

$$\inf_{0\le t\le s}\exp(-X_t) \tag{12.66}$$

has an uncertainty distribution

$$\Psi(x) = 1 - \inf_{0\le t\le s}\Phi_t(-\ln x). \tag{12.67}$$

Exercise 12.15: Let X_t be a sample-continuous and positive independent increment process with continuous uncertainty distribution $\Phi_t(x)$. Show that the supremum

$$\sup_{0\le t\le s}\frac{1}{X_t} \tag{12.68}$$

has an uncertainty distribution

$$\Psi(x) = 1 - \sup_{0\le t\le s}\Phi_t\left(\frac{1}{x}\right); \tag{12.69}$$

and the infimum

$$\inf_{0\le t\le s}\frac{1}{X_t} \tag{12.70}$$

has an uncertainty distribution

$$\Psi(x) = 1 - \inf_{0\le t\le s}\Phi_t\left(\frac{1}{x}\right). \tag{12.71}$$

12.7 First Hitting Time

Definition 12.11 *Let X_t be an uncertain process and let z be a given level. Then the uncertain variable*

$$\tau_z = \inf\left\{t\ge 0 \;\middle|\; X_t = z\right\} \tag{12.72}$$

is called the first hitting time that X_t reaches the level z.

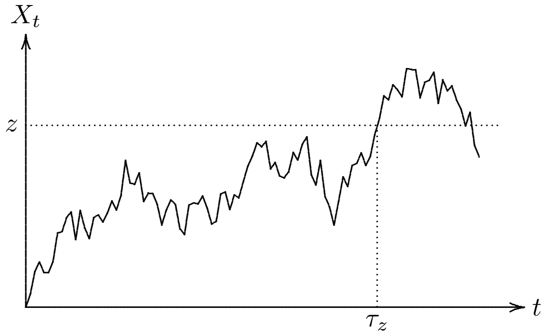


Figure 12.5: First Hitting Time

Theorem 12.19 *Let X_t be an uncertain process and let z be a given level. Then the first hitting time τ_z that X_t reaches the level z has an uncertainty distribution,*

$$\Upsilon(s) = \begin{cases} \mathcal{M}\left\{\sup\limits_{0 \le t \le s} X_t \ge z\right\}, & \text{if } X_0 < z \\ \mathcal{M}\left\{\inf\limits_{0 \le t \le s} X_t \le z\right\}, & \text{if } X_0 > z. \end{cases} \tag{12.73}$$

Proof: When $X_0 < z$, it follows from the definition of first hitting time that

$$\tau_z \le s \text{ if and only if } \sup_{0 \le t \le s} X_t \ge z.$$

Thus the uncertainty distribution of τ_z is

$$\Upsilon(s) = \mathcal{M}\{\tau_z \le s\} = \mathcal{M}\left\{\sup_{0 \le t \le s} X_t \ge z\right\}.$$

When $X_0 > z$, it follows from the definition of first hitting time that

$$\tau_z \le s \text{ if and only if } \inf_{0 \le t \le s} X_t \le z.$$

Thus the uncertainty distribution of τ_z is

$$\Upsilon(s) = \mathcal{M}\{\tau_z \le s\} = \mathcal{M}\left\{\inf_{0 \le t \le s} X_t \le z\right\}.$$

The theorem is verified.

Theorem 12.20 *(Liu [135]) Let X_t be a sample-continuous independent increment process with continuous uncertainty distribution $\Phi_t(x)$. If f is a* *strictly increasing function and z is a given level, then the first hitting time τ_z that $f(X_t)$ reaches the level z has an uncertainty distribution,*

$$\Upsilon(s) = \begin{cases} 1 - \inf\limits_{0 \le t \le s} \Phi_t(f^{-1}(z)), & \text{if } z > f(X_0) \\ \sup\limits_{0 \le t \le s} \Phi_t(f^{-1}(z)), & \text{if } z < f(X_0). \end{cases} \tag{12.74}$$

Proof: Note that X_t is a sample-continuous independent increment process and f is a strictly increasing function. When $z > f(X_0)$, it follows from the extreme value theorem that

$$\Upsilon(s) = \mathcal{M}\{\tau_z \le s\} = \mathcal{M}\left\{\sup_{0 \le t \le s} f(X_t) \ge z\right\} = 1 - \inf_{0 \le t \le s} \Phi_t(f^{-1}(z)).$$

When $z < f(X_0)$, it follows from the extreme value theorem that

$$\Upsilon(s) = \mathcal{M}\{\tau_z \le s\} = \mathcal{M}\left\{\inf_{0 \le t \le s} f(X_t) \le z\right\} = \sup_{0 \le t \le s} \Phi_t(f^{-1}(z)).$$

The theorem is verified.

Theorem 12.21 *(Liu [135]) Let X_t be a sample-continuous independent increment process with continuous uncertainty distribution $\Phi_t(x)$. If f is a strictly decreasing function and z is a given level, then the first hitting time τ_z that $f(X_t)$ reaches the level z has an uncertainty distribution,*

$$\Upsilon(s) = \begin{cases} \sup\limits_{0 \le t \le s} \Phi_t(f^{-1}(z)), & \text{if } z > f(X_0) \\ 1 - \inf\limits_{0 \le t \le s} \Phi_t(f^{-1}(z)), & \text{if } z < f(X_0). \end{cases} \tag{12.75}$$

Proof: Note that X_t is an independent increment process and f is a strictly decreasing function. When $z > f(X_0)$, it follows from the extreme value theorem that

$$\Upsilon(s) = \mathcal{M}\{\tau_z \le s\} = \mathcal{M}\left\{\sup_{0 \le t \le s} f(X_t) \ge z\right\} = \sup_{0 \le t \le s} \Phi_t(f^{-1}(z)).$$

When $z < f(X_0)$, it follows from the extreme value theorem that

$$\Upsilon(s) = \mathcal{M}\{\tau_z \le s\} = \mathcal{M}\left\{\inf_{0 \le t \le s} f(X_t) \le z\right\} = 1 - \inf_{0 \le t \le s} \Phi_t(f^{-1}(z)).$$

The theorem is verified.

12.8 Time Integral

This section will give a definition of time integral that is an integral of uncertain process with respect to time.

Definition 12.12 *(Liu [123]) Let X_t be an uncertain process. For any partition of closed interval $[a,b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|. \tag{12.76}$$

Then the time integral of X_t with respect to t is

$$\int_a^b X_t \mathrm{d}t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (t_{i+1} - t_i) \tag{12.77}$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be time integrable.

Since X_t is an uncertain variable at each time t, the limit in (12.77) is also an uncertain variable provided that the limit exists almost surely and is finite. Hence an uncertain process X_t is time integrable if and only if the limit in (12.77) is an uncertain variable.

Theorem 12.22 *If X_t is a sample-continuous uncertain process on $[a,b]$, then it is time integrable on $[a,b]$.*

Proof: Let $a = t_1 < t_2 < \cdots < t_{k+1} = b$ be a partition of the closed interval $[a,b]$. Since the uncertain process X_t is sample-continuous, almost all sample paths are continuous functions with respect to t. Hence the limit

$$\lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(t_{i+1} - t_i)$$

exists almost surely and is finite. On the other hand, since X_t is an uncertain variable at each time t, the above limit is also a measurable function. Hence the limit is an uncertain variable and then X_t is time integrable.

Theorem 12.23 *If X_t is a time integrable uncertain process on $[a,b]$, then it is time integrable on each subinterval of $[a,b]$. Moreover, if $c \in [a,b]$, then*

$$\int_a^b X_t \mathrm{d}t = \int_a^c X_t \mathrm{d}t + \int_c^b X_t \mathrm{d}t. \tag{12.78}$$

Proof: Let $[a',b']$ be a subinterval of $[a,b]$. Since X_t is a time integrable uncertain process on $[a,b]$, for any partition

$$a = t_1 < \cdots < t_m = a' < t_{m+1} < \cdots < t_n = b' < t_{n+1} < \cdots < t_{k+1} = b,$$

the limit

$$\lim_{\Delta\to 0}\sum_{i=1}^{k} X_{t_i}(t_{i+1}-t_i)$$

exists almost surely and is finite. Thus the limit

$$\lim_{\Delta\to 0}\sum_{i=m}^{n-1} X_{t_i}(t_{i+1}-t_i)$$

exists almost surely and is finite. Hence X_t is time integrable on the subinterval $[a', b']$. Next, for the partition

$$a = t_1 < \cdots < t_m = c < t_{m+1} < \cdots < t_{k+1} = b,$$

we have

$$\sum_{i=1}^{k} X_{t_i}(t_{i+1}-t_i) = \sum_{i=1}^{m-1} X_{t_i}(t_{i+1}-t_i) + \sum_{i=m}^{k} X_{t_i}(t_{i+1}-t_i).$$

Note that

$$\int_a^b X_t \mathrm{d}t = \lim_{\Delta\to 0}\sum_{i=1}^{k} X_{t_i}(t_{i+1}-t_i),$$

$$\int_a^c X_t \mathrm{d}t = \lim_{\Delta\to 0}\sum_{i=1}^{m-1} X_{t_i}(t_{i+1}-t_i),$$

$$\int_c^b X_t \mathrm{d}t = \lim_{\Delta\to 0}\sum_{i=m}^{k} X_{t_i}(t_{i+1}-t_i).$$

Hence the equation (12.78) is proved.

Theorem 12.24 *(Linearity of Time Integral) Let X_t and Y_t be time integrable uncertain processes on $[a, b]$, and let α and β be real numbers. Then*

$$\int_a^b (\alpha X_t + \beta Y_t)\mathrm{d}t = \alpha\int_a^b X_t \mathrm{d}t + \beta\int_a^b Y_t \mathrm{d}t. \tag{12.79}$$

Proof: Let $a = t_1 < t_2 < \cdots < t_{k+1} = b$ be a partition of the closed interval $[a, b]$. It follows from the definition of time integral that

$$\int_a^b (\alpha X_t + \beta Y_t)\mathrm{d}t = \lim_{\Delta\to 0}\sum_{i=1}^{k}(\alpha X_{t_i} + \beta Y_{t_i})(t_{i+1}-t_i)$$

$$= \lim_{\Delta\to 0}\alpha\sum_{i=1}^{k} X_{t_i}(t_{i+1}-t_i) + \lim_{\Delta\to 0}\beta\sum_{i=1}^{k} Y_{t_i}(t_{i+1}-t_i)$$

$$= \alpha\int_a^b X_t \mathrm{d}t + \beta\int_a^b Y_t \mathrm{d}t.$$

Hence the equation (12.79) is proved.

12.9 Bibliographic Notes

The study of uncertain process was started by Liu [123] in 2008 for modeling the evolution of uncertain phenomena. In order to describe uncertain process, Liu [139] proposed the concepts of uncertainty distribution and inverse uncertainty distribution. In addition, independence concept of uncertain processes was also introduced by Liu [139].

Independent increment process was initialized by Liu [123], and a sufficient and necessary condition was proved by Liu [139] for its inverse uncertainty distribution. In addition, Liu [135] presented an extreme value theorem and obtained the uncertainty distribution of first hitting time of independent increment process.

Stationary independent increment process was initialized by Liu [123], and its inverse uncertainty distribution was investigated by Liu [139]. Furthermore, Liu [129] showed that the expected value is a linear function of time, and Chen [17] verified that the variance is proportional to the square of time.

Chapter 13

Uncertain Renewal Process

Uncertain renewal process is an uncertain process in which events occur continuously and independently of one another in uncertain times. This chapter will introduce uncertain renewal process, renewal reward process, and alternating renewal process. This chapter will also provide block replacement policy, age replacement policy, and an uncertain insurance model.

13.1 Uncertain Renewal Process

Definition 13.1 *(Liu [123]) Let $\xi_1, \xi_2, \cdots$ be iid uncertain interarrival times. Define $S_0 = 0$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ for $n \ge 1$. Then the uncertain process*

$$N_t = \max_{n \ge 0} \{n \mid S_n \le t\} \tag{13.1}$$

is called an uncertain renewal process.

It is clear that S_n is a stationary independent increment process with respect to n. Since $\xi_1, \xi_2, \cdots$ denote the interarrival times of successive events, S_n can be regarded as the waiting time until the occurrence of the nth event. In this case, the renewal process N_t is the number of renewals in $(0, t]$. Note that N_t is not sample-continuous, but each sample path of N_t is a right-continuous and increasing step function taking only nonnegative integer values. Furthermore, since the interarrival times are always assumed to be positive uncertain variables, the size of each jump of N_t is always 1. In other words, N_t has at most one renewal at each time. In particular, N_t does not jump at time 0.

Theorem 13.1 *(Fundamental Relationship) Let N_t be a renewal process with uncertain interarrival times $\xi_1, \xi_2, \cdots,$ and $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$.*

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5_14

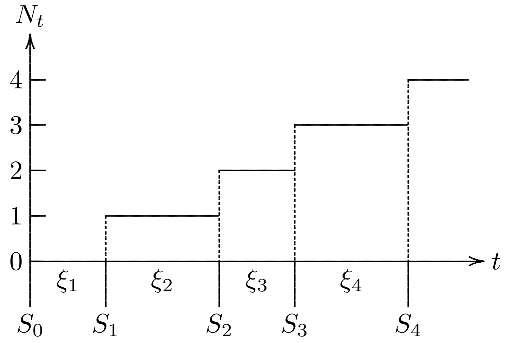


Figure 13.1: A Sample Path of Renewal Process. Reprinted from Liu [129].

Then we have

$$N_t \ge n \Leftrightarrow S_n \le t \tag{13.2}$$

for any time t and integer n. Furthermore, we also have

$$N_t \le n \Leftrightarrow S_{n+1} > t. \tag{13.3}$$

It follows from the fundamental relationship that $N_t \ge n$ is equivalent to $S_n \le t$. Thus we immediately have

$$\mathcal{M}\{N_t \ge n\} = \mathcal{M}\{S_n \le t\}. \tag{13.4}$$

Since $N_t \le n$ is equivalent to $S_{n+1} > t$, by using the duality axiom, we also have

$$\mathcal{M}\{N_t \le n\} = 1 - \mathcal{M}\{S_{n+1} \le t\}. \tag{13.5}$$

Theorem 13.2 *(Liu [129]) Let N_t be a renewal process with uncertain interarrival times $\xi_1, \xi_2, \cdots$ If those interarrival times have a common uncertainty distribution Φ, then N_t has an uncertainty distribution*

$$\Upsilon_t(x) = 1 - \Phi\left(\frac{t}{\lfloor x \rfloor + 1}\right), \quad \forall x \ge 0 \tag{13.6}$$

where $\lfloor x \rfloor$ represents the maximal integer less than or equal to x.

Proof: Note that S_{n+1} has an uncertainty distribution $\Phi(x/(n+1))$. It follows from (13.5) that

$$\mathcal{M}\{N_t \le n\} = 1 - \mathcal{M}\{S_{n+1} \le t\} = 1 - \Phi\left(\frac{t}{n+1}\right).$$

Since N_t takes integer values, for any $x \ge 0$, we have

$$\Upsilon_t(x) = \mathcal{M}\{N_t \le x\} = \mathcal{M}\{N_t \le \lfloor x \rfloor\} = 1 - \Phi\left(\frac{t}{\lfloor x \rfloor + 1}\right).$$

The theorem is verified.

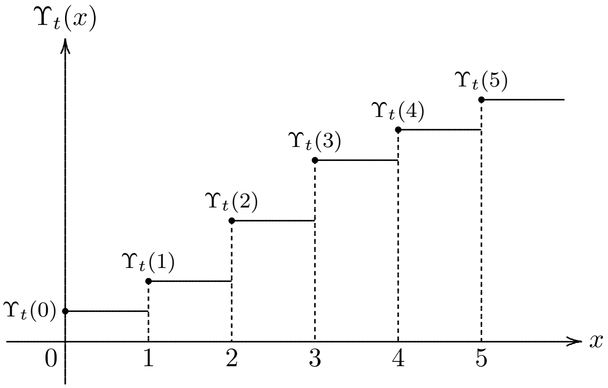


Figure 13.2: Uncertainty Distribution $\Upsilon_t(x)$ of Renewal Process N_t. Reprinted from Liu [129].

Theorem 13.3 *(Liu [129]) Let N_t be a renewal process with uncertain interarrival times $\xi_1, \xi_2, \cdots$ Then the average renewal number*

$$\frac{N_t}{t} \to \frac{1}{\xi_1} \tag{13.7}$$

in the sense of convergence in distribution as $t \to \infty$.

Proof: The uncertainty distribution Υ_t of N_t has been given by Theorem 13.2 as follows,

$$\Upsilon_t(x) = 1 - \Phi\left(\frac{t}{\lfloor x \rfloor + 1}\right)$$

where Φ is the uncertainty distribution of ξ_1. It follows from the operational law that the uncertainty distribution of N_t/t is

$$\Psi_t(x) = 1 - \Phi\left(\frac{t}{\lfloor tx \rfloor + 1}\right)$$

where $\lfloor tx \rfloor$ represents the maximal integer less than or equal to tx. Thus at each continuity point x of $1 - \Phi(1/x)$, we have

$$\lim_{t\to\infty} \Psi_t(x) = 1 - \Phi\left(\frac{1}{x}\right)$$

which is just the uncertainty distribution of $1/\xi_1$. Hence N_t/t converges in distribution to $1/\xi_1$ as $t \to \infty$.

Theorem 13.4 *(Liu [129], Elementary Renewal Theorem) Let N_t be a renewal process with uncertain interarrival times $\xi_1, \xi_2, \cdots$ If $E[1/\xi_1]$ exists, then*

$$\lim_{t\to\infty} \frac{E[N_t]}{t} = E\left[\frac{1}{\xi_1}\right]. \tag{13.8}$$

If those interarrival times have a common uncertainty distribution Φ, *then*

$$\lim_{t\to\infty}\frac{E[N_t]}{t}=\int_0^{+\infty}\Phi\left(\frac{1}{x}\right)\mathrm{d}x. \tag{13.9}$$

If the uncertainty distribution Φ *is regular, then*

$$\lim_{t\to\infty}\frac{E[N_t]}{t}=\int_0^{1}\frac{1}{\Phi^{-1}(\alpha)}\mathrm{d}\alpha. \tag{13.10}$$

Proof: Write the uncertainty distributions of N_t/t and $1/\xi_1$ by $\Psi_t(x)$ and $G(x)$, respectively. Theorem 13.3 says that $\Psi_t(x)\to G(x)$ as $t\to\infty$ at each continuity point x of $G(x)$. Note that $\Psi_t(x)\ge G(x)$. It follows from Lebesgue dominated convergence theorem and the existence of $E[1/\xi_1]$ that

$$\lim_{t\to\infty}\frac{E[N_t]}{t}=\lim_{t\to\infty}\int_0^{+\infty}(1-\Psi_t(x))\mathrm{d}x=\int_0^{+\infty}(1-G(x))\mathrm{d}x=E\left[\frac{1}{\xi_1}\right].$$

Since $1/\xi_1$ has an uncertainty distribution $1-\Phi(1/x)$, we have

$$\lim_{t\to\infty}\frac{E[N_t]}{t}=E\left[\frac{1}{\xi_1}\right]=\int_0^{+\infty}\Phi\left(\frac{1}{x}\right)\mathrm{d}x.$$

Furthermore, since $1/\xi$ has an inverse uncertainty distribution

$$G^{-1}(\alpha)=\frac{1}{\Phi^{-1}(1-\alpha)},$$

we get

$$E\left[\frac{1}{\xi}\right]=\int_0^1\frac{1}{\Phi^{-1}(1-\alpha)}\mathrm{d}\alpha=\int_0^1\frac{1}{\Phi^{-1}(\alpha)}\mathrm{d}\alpha.$$

The theorem is proved.

Exercise 13.1: A renewal process N_t is called *linear* if $\xi_1,\xi_2,\cdots$ are iid linear uncertain variables $\mathcal{L}(a,b)$ with $a>0$. Show that

$$\lim_{t\to\infty}\frac{E[N_t]}{t}=\frac{\ln b-\ln a}{b-a}. \tag{13.11}$$

Exercise 13.2: A renewal process N_t is called *zigzag* if $\xi_1,\xi_2,\cdots$ are iid zigzag uncertain variables $\mathcal{Z}(a,b,c)$ with $a>0$. Show that

$$\lim_{t\to\infty}\frac{E[N_t]}{t}=\frac{1}{2}\left(\frac{\ln b-\ln a}{b-a}+\frac{\ln c-\ln b}{c-b}\right). \tag{13.12}$$

Exercise 13.3: A renewal process N_t is called *lognormal* if $\xi_1,\xi_2,\cdots$ are iid lognormal uncertain variables $\mathcal{LOGN}(e,\sigma)$. Show that

$$\lim_{t\to\infty}\frac{E[N_t]}{t}=\begin{cases}\sqrt{3}\sigma\exp(-e)\csc(\sqrt{3}\sigma), & \text{if } \sigma<\pi/\sqrt{3}\\ +\infty, & \text{if } \sigma\ge\pi/\sqrt{3}.\end{cases} \tag{13.13}$$

13.2 Block Replacement Policy

Block replacement policy means that an element is always replaced at failure or periodically with time s. Assume that the lifetimes of elements are iid uncertain variables $\xi_1, \xi_2, \cdots$ with a common uncertainty distribution Φ. Then the replacement times form an uncertain renewal process N_t. Let a denote the "failure replacement" cost of replacing an element when it fails earlier than s, and b the "planned replacement" cost of replacing an element at planned time s. Note that $a > b > 0$ is always assumed. It is clear that the cost of one period is $aN_s + b$ and the average cost is

$$\frac{aN_s + b}{s}. \tag{13.14}$$

Theorem 13.5 *(Yao [250]) Assume the lifetimes of elements are iid uncertain variables $\xi_1, \xi_2, \cdots$ with a common uncertainty distribution Φ, and N_t is the uncertain renewal process representing the replacement times. Then the average cost has an expected value*

$$E\left[\frac{aN_s + b}{s}\right] = \frac{1}{s}\left(a\sum_{n=1}^{\infty}\Phi\left(\frac{s}{n}\right) + b\right). \tag{13.15}$$

Proof: Note that the uncertainty distribution of N_t is a step function. It follows from Theorem 13.2 that

$$E[N_s] = \int_0^{+\infty}\Phi\left(\frac{s}{\lfloor x\rfloor + 1}\right)\mathrm{d}x = \sum_{n=1}^{\infty}\Phi\left(\frac{s}{n}\right).$$

Thus (13.15) is verified by

$$E\left[\frac{aN_s + b}{s}\right] = \frac{aE[N_s] + b}{s}. \tag{13.16}$$

Finally, please note that

$$\lim_{s\downarrow 0} E\left[\frac{aN_s + b}{s}\right] = +\infty, \tag{13.17}$$

$$\lim_{s\to\infty} E\left[\frac{aN_s + b}{s}\right] = a\int_0^{+\infty}\Phi\left(\frac{1}{x}\right)\mathrm{d}x. \tag{13.18}$$

What is the optimal time s?

When the block replacement policy is accepted, one problem is concerned with finding an optimal time s in order to minimize the average cost, i.e.,

$$\min_s \frac{1}{s}\left(a\sum_{n=1}^{\infty}\Phi\left(\frac{s}{n}\right) + b\right). \tag{13.19}$$

13.3 Renewal Reward Process

Let $(\xi_1, \eta_1), (\xi_2, \eta_2), \cdots$ be a sequence of pairs of uncertain variables. We shall interpret η_i as the rewards (or costs) associated with the i-th interarrival times ξ_i for $i = 1, 2, \cdots$, respectively.

Definition 13.2 *(Liu [129]) Let $\xi_1, \xi_2, \cdots$ be iid uncertain interarrival times, and let $\eta_1, \eta_2, \cdots$ be iid uncertain rewards. Assume that $(\xi_1, \xi_2, \cdots)$ and $(\eta_1, \eta_2, \cdots)$ are independent uncertain vectors. Then*

$$R_t = \sum_{i=1}^{N_t} \eta_i \tag{13.20}$$

is called a renewal reward process, where N_t is the renewal process with uncertain interarrival times $\xi_1, \xi_2, \cdots$

A renewal reward process R_t denotes the total reward earned by time t. In addition, if $\eta_i \equiv 1$, then R_t degenerates to a renewal process N_t. Please also note that $R_t = 0$ whenever $N_t = 0$.

Theorem 13.6 *(Liu [129]) Let R_t be a renewal reward process with uncertain interarrival times $\xi_1, \xi_2, \cdots$ and uncertain rewards $\eta_1, \eta_2, \cdots$ Assume those interarrival times and rewards have uncertainty distributions Φ and Ψ, respectively. Then R_t has an uncertainty distribution*

$$\Upsilon_t(x) = \max_{k \ge 0} \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{x}{k}\right). \tag{13.21}$$

Here we set $x/k = +\infty$ and $\Psi(x/k) = 1$ when $k = 0$.

Proof: It follows from the definition of renewal reward process that the renewal process N_t is independent of uncertain rewards $\eta_1, \eta_2, \cdots$, and R_t has an uncertainty distribution

$$\begin{aligned}
\Upsilon_t(x) &= \mathcal{M}\left\{\sum_{i=1}^{N_t} \eta_i \le x\right\} = \mathcal{M}\left\{\bigcup_{k=0}^{\infty} (N_t = k) \cap \sum_{i=1}^{k} \eta_i \le x\right\} \\
&= \mathcal{M}\left\{\bigcup_{k=0}^{\infty} (N_t = k) \cap \left(\eta_1 \le \frac{x}{k}\right)\right\} \quad \text{(this is a polyrectangle)} \\
&= \max_{k \ge 0} \mathcal{M}\left\{(N_t \le k) \cap \left(\eta_1 \le \frac{x}{k}\right)\right\} \quad \text{(polyrectangular theorem)} \\
&= \max_{k \ge 0} \mathcal{M}\{N_t \le k\} \wedge \mathcal{M}\left\{\eta_1 \le \frac{x}{k}\right\} \quad \text{(independence)} \\
&= \max_{k \ge 0} \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{x}{k}\right).
\end{aligned}$$

The theorem is proved.

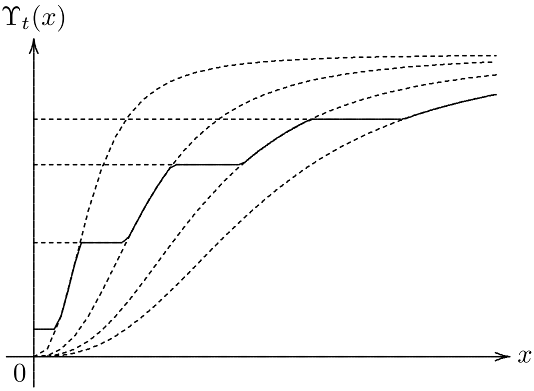


Figure 13.3: Uncertainty Distribution $\Upsilon_t(x)$ of Renewal Reward Process R_t in which the dashed horizontal lines are $1 - \Phi(t/(k+1))$ and the dashed curves are $\Psi(x/k)$ for $k = 0, 1, 2, \cdots$ Reprinted from Liu [129].

Theorem 13.7 *(Liu [129]) Assume that R_t is a renewal reward process with uncertain interarrival times $\xi_1, \xi_2, \cdots$ and uncertain rewards $\eta_1, \eta_2, \cdots$ Then the reward rate*

$$\frac{R_t}{t} \to \frac{\eta_1}{\xi_1} \tag{13.22}$$

in the sense of convergence in distribution as $t \to \infty$.

Proof: It follows from Theorem 13.6 that the uncertainty distribution of R_t is

$$\Upsilon_t(x) = \max_{k \ge 0} \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{x}{k}\right).$$

Then R_t/t has an uncertainty distribution

$$\Psi_t(x) = \max_{k \ge 0} \left(1 - \Phi\left(\frac{t}{k+1}\right)\right) \wedge \Psi\left(\frac{tx}{k}\right).$$

When $t \to \infty$, we have

$$\Psi_t(x) \to \sup_{y \ge 0} (1 - \Phi(y)) \wedge \Psi(xy)$$

which is just the uncertainty distribution of η_1/ξ_1. Hence R_t/t converges in distribution to η_1/ξ_1 as $t \to \infty$.

Theorem 13.8 *(Liu [129], Renewal Reward Theorem) Assume that R_t is a renewal reward process with uncertain interarrival times $\xi_1, \xi_2, \cdots$ and uncertain rewards $\eta_1, \eta_2, \cdots$ If $E[\eta_1/\xi_1]$ exists, then*

$$\lim_{t \to \infty} \frac{E[R_t]}{t} = E\left[\frac{\eta_1}{\xi_1}\right]. \tag{13.23}$$

If those interarrival times and rewards have regular uncertainty distributions Φ and Ψ, respectively, then

$$\lim_{t\to\infty} \frac{E[R_t]}{t} = \int_0^1 \frac{\Psi^{-1}(\alpha)}{\Phi^{-1}(1-\alpha)}\mathrm{d}\alpha. \tag{13.24}$$

Proof: It follows from Theorem 13.6 that R_t/t has an uncertainty distribution

$$F_t(x) = \max_{k\ge 0}\left(1-\Phi\left(\frac{t}{k+1}\right)\right)\wedge\Psi\left(\frac{tx}{k}\right)$$

and η_1/ξ_1 has an uncertainty distribution

$$G(x) = \sup_{y\ge 0}(1-\Phi(y))\wedge\Psi(xy).$$

Note that $F_t(x)\to G(x)$ and $F_t(x)\ge G(x)$. It follows from Lebesgue dominated convergence theorem and the existence of $E[\eta_1/\xi_1]$ that

$$\lim_{t\to\infty}\frac{E[R_t]}{t} = \lim_{t\to\infty}\int_0^{+\infty}(1-F_t(x))\mathrm{d}x = \int_0^{+\infty}(1-G(x))\mathrm{d}x = E\left[\frac{\eta_1}{\xi_1}\right].$$

Finally, since η_1/ξ_1 has an inverse uncertainty distribution

$$G^{-1}(\alpha) = \frac{\Psi^{-1}(\alpha)}{\Phi^{-1}(1-\alpha)},$$

the equation (13.24) is obtained.

13.4 Uncertain Insurance Model

Liu [135] assumed that a is the initial capital of an insurance company, b is the premium rate, bt is the total income up to time t, and the uncertain claim process is a renewal reward process

$$R_t = \sum_{i=1}^{N_t}\eta_i \tag{13.25}$$

with iid uncertain interarrival times $\xi_1,\xi_2,\cdots$ and iid uncertain claim amounts $\eta_1,\eta_2,\cdots$ Then the capital of the insurance company at time t is

$$Z_t = a + bt - R_t \tag{13.26}$$

and Z_t is called an *insurance risk process.*

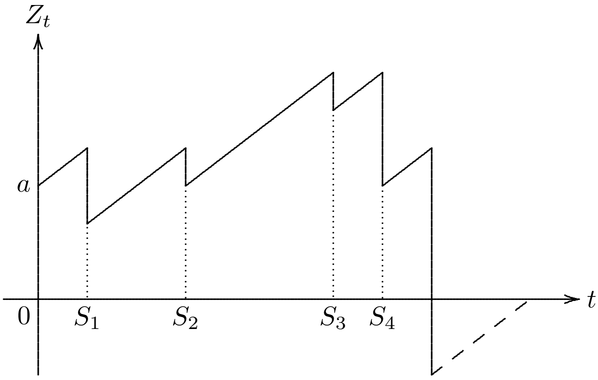


Figure 13.4: An Insurance Risk Process

Ruin Index

Ruin index is the uncertain measure that the capital of the insurance company becomes negative.

Definition 13.3 *(Liu [135]) Let Z_t be an insurance risk process. Then the ruin index is defined as the uncertain measure that Z_t eventually becomes negative, i.e.,*

$$Ruin = \mathcal{M}\left\{\inf_{t\ge 0} Z_t < 0\right\}. \tag{13.27}$$

It is clear that the ruin index is a special case of the risk index in the sense of Liu [128].

Theorem 13.9 *(Liu [135], Ruin Index Theorem) Let $Z_t = a + bt - R_t$ be an insurance risk process where a and b are positive numbers, and R_t is a renewal reward process with iid uncertain interarrival times $\xi_1, \xi_2, \cdots$ and iid uncertain claim amounts $\eta_1, \eta_2, \cdots$ If ξ_1 and η_1 have continuous uncertainty distributions Φ and Ψ, respectively, then the ruin index is*

$$Ruin = \max_{k\ge 1}\sup_{x\ge 0}\Phi\left(\frac{x-a}{kb}\right)\wedge\left(1-\Psi\left(\frac{x}{k}\right)\right). \tag{13.28}$$

Proof: For each positive integer k, it is clear that the arrival time of the kth claim is

$$S_k = \xi_1 + \xi_2 + \cdots + \xi_k$$

whose uncertainty distribution is $\Phi(s/k)$. Define an uncertain process indexed by k as follows,

$$Y_k = a + bS_k - (\eta_1 + \eta_2 + \cdots + \eta_k).$$

It is easy to verify that Y_k is an independent increment process with respect to k. In addition, Y_k is just the capital at the arrival time S_k and has an uncertainty distribution

$$F_k(z) = \sup_{x\ge 0} \Phi\left(\frac{z+x-a}{kb}\right) \wedge \left(1-\Psi\left(\frac{x}{k}\right)\right).$$

Since a ruin occurs only at the arrival times, we have

$$Ruin = \mathcal{M}\left\{\inf_{t\ge 0} Z_t < 0\right\} = \mathcal{M}\left\{\min_{k\ge 1} Y_k < 0\right\}.$$

It follows from the extreme value theorem that

$$Ruin = \max_{k\ge 1} F_k(0) = \max_{k\ge 1}\sup_{x\ge 0} \Phi\left(\frac{x-a}{kb}\right) \wedge \left(1-\Psi\left(\frac{x}{k}\right)\right).$$

The theorem is proved.

Ruin Time

Definition 13.4 *(Liu [135]) Let Z_t be an insurance risk process. Then the ruin time is determined by*

$$\tau = \inf\left\{t\ge 0 \;\middle|\; Z_t < 0\right\}. \tag{13.29}$$

If $Z_t \ge 0$ for all $t\ge 0$, then we define $\tau = +\infty$. Note that the ruin time is just the first hitting time that the total capital Z_t becomes negative. Since $\inf_{t\ge 0} Z_t < 0$ if and only if $\tau < +\infty$, the relation between ruin index and ruin time is

$$Ruin = \mathcal{M}\left\{\inf_{t\ge 0} Z_t < 0\right\} = \mathcal{M}\{\tau < +\infty\}.$$

Theorem 13.10 *(Yao [257]) Let $Z_t = a + bt - R_t$ be an insurance risk process where a and b are positive numbers, and R_t is a renewal reward process with iid uncertain interarrival times $\xi_1, \xi_2, \cdots$ and iid uncertain claim amounts $\eta_1, \eta_2, \cdots$ If ξ_1 and η_1 have regular uncertainty distributions Φ and Ψ, respectively, then the ruin time has an uncertainty distribution*

$$\Upsilon(t) = \max_{k\ge 1}\sup_{x\le t} \Phi\left(\frac{x}{k}\right) \wedge \left(1-\Psi\left(\frac{a+bx}{k}\right)\right). \tag{13.30}$$

Proof: For each positive integer k, let us write $S_k = \xi_1+\xi_2+\cdots+\xi_k$, $Y_k = a + bS_k - (\eta_1+\eta_2+\cdots+\eta_k)$ and

$$\alpha_k = \sup_{x\le t} \Phi\left(\frac{x}{k}\right) \wedge \left(1-\Psi\left(\frac{a+bx}{k}\right)\right).$$

Then

$$\alpha_k = \sup\left\{\alpha \,|\, k\Phi^{-1}(\alpha) \le t\right\} \wedge \sup\left\{\alpha \,|\, a + k\Phi^{-1}(\alpha) - k\Psi^{-1}(1-\alpha) < 0\right\}.$$

On the one hand, it follows from the definition of the ruin time τ that

$$\begin{aligned}
\mathcal{M}\{\tau \le t\} &= \mathcal{M}\left\{\inf_{0\le s\le t} Z_s < 0\right\} = \mathcal{M}\left\{\bigcup_{k=1}^{\infty}(S_k \le t,\, Y_k < 0)\right\} \\
&= \mathcal{M}\left\{\bigcup_{k=1}^{\infty}\left(\sum_{i=1}^{k}\xi_i \le t,\, a + b\sum_{i=1}^{k}\xi_i - \sum_{i=1}^{k}\eta_i < 0\right)\right\} \\
&\ge \mathcal{M}\left\{\bigcup_{k=1}^{\infty}\bigcap_{i=1}^{k}(\xi_i \le \Phi^{-1}(\alpha_k)) \cap (\eta_i > \Psi^{-1}(1-\alpha_k))\right\} \\
&\ge \bigvee_{k=1}^{\infty}\mathcal{M}\left\{\bigcap_{i=1}^{k}(\xi_i \le \Phi^{-1}(\alpha_k)) \cap (\eta_i > \Psi^{-1}(1-\alpha_k))\right\} \\
&= \bigvee_{k=1}^{\infty}\bigwedge_{i=1}^{k}\mathcal{M}\left\{(\xi_i \le \Phi^{-1}(\alpha_k)) \cap (\eta_i > \Psi^{-1}(1-\alpha_k))\right\} \\
&= \bigvee_{k=1}^{\infty}\bigwedge_{i=1}^{k}\mathcal{M}\left\{\xi_i \le \Phi^{-1}(\alpha_k)\right\} \wedge \mathcal{M}\left\{\eta_i > \Psi^{-1}(1-\alpha_k)\right\} \\
&= \bigvee_{k=1}^{\infty}\bigwedge_{i=1}^{k}\alpha_k \wedge \alpha_k = \bigvee_{k=1}^{\infty}\alpha_k.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\mathcal{M}\{\tau \le t\} &= \mathcal{M}\left\{\bigcup_{k=1}^{\infty}\left(\sum_{i=1}^{k}\xi_i \le t,\, a + b\sum_{i=1}^{k}\xi_i - \sum_{i=1}^{k}\eta_i < 0\right)\right\} \\
&\le \mathcal{M}\left\{\bigcup_{k=1}^{\infty}\bigcup_{i=1}^{k}(\xi_i \le \Phi^{-1}(\alpha_k)) \cup (\eta_i > \Psi^{-1}(1-\alpha_k))\right\} \\
&= \mathcal{M}\left\{\bigcup_{i=1}^{\infty}\bigcup_{k=i}^{\infty}(\xi_i \le \Phi^{-1}(\alpha_k)) \cup (\eta_i > \Psi^{-1}(1-\alpha_k))\right\} \\
&\le \mathcal{M}\left\{\bigcup_{i=1}^{\infty}\left(\xi_i \le \bigvee_{k=i}^{\infty}\Phi^{-1}(\alpha_k)\right) \cup \left(\eta_i > \bigwedge_{k=i}^{\infty}\Psi^{-1}(1-\alpha_k)\right)\right\} \\
&= \bigvee_{i=1}^{\infty}\mathcal{M}\left\{\xi_i \le \bigvee_{k=i}^{\infty}\Phi^{-1}(\alpha_k)\right\} \vee \mathcal{M}\left\{\eta_i > \bigwedge_{k=i}^{\infty}\Psi^{-1}(1-\alpha_k)\right\} \\
&= \bigvee_{i=1}^{\infty}\bigvee_{k=i}^{\infty}\alpha_k \vee \left(1 - \bigwedge_{k=i}^{\infty}(1-\alpha_k)\right) = \bigvee_{k=1}^{\infty}\alpha_k.
\end{aligned}$$

Thus we obtain

$$\mathcal{M}\{\tau \le t\} = \bigvee_{k=1}^{\infty} \alpha_k$$

and the theorem is verified.

13.5 Age Replacement Policy

Age replacement means that an element is always replaced at failure or at an age s. Assume that the lifetimes of the elements are iid uncertain variables $\xi_1, \xi_2, \cdots$ with a common uncertainty distribution Φ. Then the actual lifetimes of the elements are iid uncertain variables

$$\xi_1 \wedge s,\ \xi_2 \wedge s,\ \cdots \tag{13.31}$$

which may generate an uncertain renewal process

$$N_t = \max_{n\ge 0}\left\{ n \;\Big|\; \sum_{i=1}^{n} (\xi_i \wedge s) \le t \right\}. \tag{13.32}$$

Let a denote the "failure replacement" cost of replacing an element when it fails earlier than s, and b the "planned replacement" cost of replacing an element at the age s. Note that $a > b > 0$ is always assumed. Define

$$f(x) = \begin{cases} a, & \text{if } x < s \\ b, & \text{if } x = s. \end{cases} \tag{13.33}$$

Then $f(\xi_i \wedge s)$ is just the cost of replacing the ith element, and the average replacement cost before the time t is

$$\frac{1}{t}\sum_{i=1}^{N_t} f(\xi_i \wedge s). \tag{13.34}$$

Theorem 13.11 *(Yao and Ralescu [245]) Assume $\xi_1, \xi_2, \cdots$ are iid uncertain lifetimes and s is a positive number. Then*

$$\frac{1}{t}\sum_{i=1}^{N_t} f(\xi_i \wedge s) \to \frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \tag{13.35}$$

in the sense of convergence in distribution as $t \to \infty$.

Proof: At first, the average replacement cost before time t may be rewritten as

$$\frac{1}{t}\sum_{i=1}^{N_t} f(\xi_i \wedge s) = \frac{\displaystyle\sum_{i=1}^{N_t} f(\xi_i \wedge s)}{\displaystyle\sum_{i=1}^{N_t} (\xi_i \wedge s)} \times \frac{\displaystyle\sum_{i=1}^{N_t} (\xi_i \wedge s)}{t}. \tag{13.36}$$

For any real number x, on the one hand, we have

$$\begin{aligned}
&\left\{\sum_{i=1}^{N_t} f(\xi_i\wedge s)/\sum_{i=1}^{N_t}(\xi_i\wedge s)\le x\right\}\\
&=\bigcup_{n=1}^{\infty}\left\{(N_t=n)\cap\left(\sum_{i=1}^{n} f(\xi_i\wedge s)/\sum_{i=1}^{n}(\xi_i\wedge s)\le x\right)\right\}\\
&\supset\bigcup_{n=1}^{\infty}\left\{(N_t=n)\cap\bigcap_{i=1}^{n}(f(\xi_i\wedge s)/(\xi_i\wedge s)\le x)\right\}\\
&\supset\bigcup_{n=1}^{\infty}\left\{(N_t=n)\cap\bigcap_{i=1}^{\infty}(f(\xi_i\wedge s)/(\xi_i\wedge s)\le x)\right\}\\
&\supset\bigcap_{i=1}^{\infty}(f(\xi_i\wedge s)/(\xi_i\wedge s)\le x)
\end{aligned}$$

and

$$\mathcal{M}\left\{\frac{\displaystyle\sum_{i=1}^{N_t} f(\xi_i\wedge s)}{\displaystyle\sum_{i=1}^{N_t}(\xi_i\wedge s)}\le x\right\}\ge\mathcal{M}\left\{\bigcap_{i=1}^{\infty}\left(\frac{f(\xi_i\wedge s)}{\xi_i\wedge s}\le x\right)\right\}=\mathcal{M}\left\{\frac{f(\xi_1\wedge s)}{\xi_1\wedge s}\le x\right\}.$$

On the other hand, we have

$$\begin{aligned}
&\left\{\sum_{i=1}^{N_t} f(\xi_i\wedge s)/\sum_{i=1}^{N_t}(\xi_i\wedge s)\le x\right\}\\
&=\bigcup_{n=1}^{\infty}\left\{(N_t=n)\cap\left(\sum_{i=1}^{n} f(\xi_i\wedge s)/\sum_{i=1}^{n}(\xi_i\wedge s)\le x\right)\right\}\\
&\subset\bigcup_{n=1}^{\infty}\left\{(N_t=n)\cap\bigcup_{i=1}^{n}(f(\xi_i\wedge s)/(\xi_i\wedge s)\le x)\right\}\\
&\subset\bigcup_{n=1}^{\infty}\left\{(N_t=n)\cap\bigcup_{i=1}^{\infty}(f(\xi_i\wedge s)/(\xi_i\wedge s)\le x)\right\}\\
&\subset\bigcup_{i=1}^{\infty}(f(\xi_i\wedge s)/(\xi_i\wedge s)\le x)
\end{aligned}$$

and

$$\mathcal{M}\left\{\frac{\sum_{i=1}^{N_t} f(\xi_i \wedge s)}{\sum_{i=1}^{N_t}(\xi_i \wedge s)} \le x\right\} \le \mathcal{M}\left\{\bigcup_{i=1}^{\infty}\left(\frac{f(\xi_i \wedge s)}{\xi_i \wedge s} \le x\right)\right\} = \mathcal{M}\left\{\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \le x\right\}.$$

Thus for any real number x, we have

$$\mathcal{M}\left\{\frac{\sum_{i=1}^{N_t} f(\xi_i \wedge s)}{\sum_{i=1}^{N_t}(\xi_i \wedge s)} \le x\right\} = \mathcal{M}\left\{\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \le x\right\}.$$

Hence

$$\frac{\sum_{i=1}^{N_t} f(\xi_i \wedge s)}{\sum_{i=1}^{N_t}(\xi_i \wedge s)} \quad \text{and} \quad \frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s}$$

are identically distributed uncertain variables. Since

$$\frac{\sum_{i=1}^{N_t}(\xi_i \wedge s)}{t} \to 1$$

as $t \to \infty$, it follows from (13.36) that (13.35) holds. The theorem is verified.

Theorem 13.12 *(Yao and Ralescu [245]) Assume* $\xi_1, \xi_2, \cdots$ *are iid uncertain lifetimes with a common continuous uncertainty distribution* Φ, *and* s *is a positive number. Then the long-run average replacement cost is*

$$\lim_{t\to\infty} E\left[\frac{1}{t}\sum_{i=1}^{N_t} f(\xi_i \wedge s)\right] = \frac{b}{s} + \frac{a-b}{s}\Phi(s) + a\int_0^s \frac{\Phi(x)}{x^2}\mathrm{d}x. \qquad (13.37)$$

Proof: Let $\Psi(x)$ be the uncertainty distribution of $f(\xi_1 \wedge s)/(\xi_1 \wedge s)$. It follows from (13.33) that $f(\xi_1 \wedge s) \ge b$ and $\xi_1 \wedge s \le s$. Thus we have

$$\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \ge \frac{b}{s}$$

almost surely. If $x < b/s$, then

$$\Psi(x) = \mathcal{M}\left\{\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \le x\right\} = 0.$$

If $b/s \le x < a/s$, then

$$\Psi(x) = \mathcal{M}\left\{\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \le x\right\} = \mathcal{M}\{\xi_1 \ge s\} = 1 - \Phi(s).$$

If $x \ge a/s$, then

$$\Psi(x) = \mathcal{M}\left\{\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \le x\right\} = \mathcal{M}\left\{\frac{a}{\xi_1} \le x\right\} = \mathcal{M}\left\{\xi_1 \ge \frac{a}{x}\right\} = 1 - \Phi\left(\frac{a}{x}\right).$$

Hence we have

$$\Psi(x) = \begin{cases} 0, & \text{if } x < b/s \\ 1 - \Phi(s), & \text{if } b/s \le x < a/s \\ 1 - \Phi(a/x), & \text{if } x \ge a/s \end{cases}$$

and

$$E\left[\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s}\right] = \int_0^{+\infty} (1 - \Psi(x))\mathrm{d}x = \frac{b}{s} + \frac{a-b}{s}\Phi(s) + a\int_0^s \frac{\Phi(x)}{x^2}\mathrm{d}x.$$

Since

$$\frac{\sum_{i=1}^{N_t}(\xi_i \wedge s)}{t} \le 1,$$

it follows from (13.36) that

$$\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t} f(\xi_i \wedge s) \le x\right\} \ge \mathcal{M}\left\{\frac{f(\xi_1 \wedge s)}{\xi \wedge s} \le x\right\}$$

for any real number x. By using the Lebesgue dominated convergence theorem, we get

$$\begin{aligned}\lim_{t\to\infty} E\left[\frac{1}{t}\sum_{i=1}^{N_t} f(\xi_i \wedge s)\right] &= \lim_{t\to\infty}\int_0^{+\infty}\left(1 - \mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t} f(\xi_i \wedge s) \le x\right\}\right)\mathrm{d}x \\ &= \int_0^{+\infty}\left(1 - \mathcal{M}\left\{\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s} \le x\right\}\right)\mathrm{d}x \\ &= E\left[\frac{f(\xi_1 \wedge s)}{\xi_1 \wedge s}\right].\end{aligned}$$

Hence the theorem is proved. Please also note that

$$\lim_{s\to 0+}\lim_{t\to\infty} E\left[\frac{1}{t}\sum_{i=1}^{N_t} f(\xi_i \wedge s)\right] = +\infty, \tag{13.38}$$

$$\lim_{s\to +\infty}\lim_{t\to\infty} E\left[\frac{1}{t}\sum_{i=1}^{N_t} f(\xi_i \wedge s)\right] = a\int_0^{+\infty} \frac{\Phi(x)}{x^2}\mathrm{d}x. \tag{13.39}$$

What is the optimal age s?

When the age replacement policy is accepted, one problem is to find the optimal age s such that the average replacement cost is minimized. That is, the optimal age s should solve

$$\min_{s\ge 0}\left(\frac{b}{s}+\frac{a-b}{s}\Phi(s)+a\int_0^s\frac{\Phi(x)}{x^2}\mathrm{d}x\right). \tag{13.40}$$

13.6 Alternating Renewal Process

Let $(\xi_1,\eta_1),(\xi_2,\eta_2),\cdots$ be a sequence of pairs of uncertain variables. We shall interpret ξ_i as the "on-times" and η_i as the "off-times" for $i=1,2,\cdots$, respectively. In this case, the i-th cycle consists of an on-time ξ_i followed by an off-time η_i.

Definition 13.5 *(Yao and Li [242]) Let $\xi_1,\xi_2,\cdots$ be iid uncertain on-times, and let $\eta_1,\eta_2,\cdots$ be iid uncertain off-times. Assume that $(\xi_1,\xi_2,\cdots)$ and $(\eta_1,\eta_2,\cdots)$ are independent uncertain vectors. Then*

$$A_t=\begin{cases} t-\displaystyle\sum_{i=1}^{N_t}\eta_i, & \text{if } \displaystyle\sum_{i=1}^{N_t}(\xi_i+\eta_i)\le t<\sum_{i=1}^{N_t}(\xi_i+\eta_i)+\xi_{N_t+1}\\ \displaystyle\sum_{i=1}^{N_t+1}\xi_i, & \text{if } \displaystyle\sum_{i=1}^{N_t}(\xi_i+\eta_i)+\xi_{N_t+1}\le t<\sum_{i=1}^{N_t+1}(\xi_i+\eta_i)\end{cases} \tag{13.41}$$

is called an alternating renewal process, where N_t is the renewal process with uncertain interarrival times $\xi_1+\eta_1,\xi_2+\eta_2,\cdots$

Note that the alternating renewal process A_t is just the total time at which the system is on up to time t. It is clear that

$$\sum_{i=1}^{N_t}\xi_i\le A_t\le\sum_{i=1}^{N_t+1}\xi_i \tag{13.42}$$

for each time t. We are interested in the limit property of the rate at which the system is on.

Theorem 13.13 *(Yao and Li [242]) Assume A_t is an alternating renewal process with uncertain on-times $\xi_1,\xi_2,\cdots$ and uncertain off-times $\eta_1,\eta_2,\cdots$ Then the availability rate*

$$\frac{A_t}{t}\to\frac{\xi_1}{\xi_1+\eta_1} \tag{13.43}$$

in the sense of convergence in distribution as $t\to\infty$.

Proof: Write the uncertainty distributions of ξ_1 and η_1 by Φ and Ψ, respectively. Then the uncertainty distribution of $\xi_1/(\xi_1+\eta_1)$ is

$$\Upsilon(x)=\sup_{y>0}\Phi(xy)\wedge(1-\Psi(y-xy)).$$

On the one hand, we have

$$\begin{aligned}
&\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\xi_i\le x\right\}\\
&=\mathcal{M}\left\{\bigcup_{k=0}^{\infty}(N_t=k)\cap\left(\frac{1}{t}\sum_{i=1}^{k}\xi_i\le x\right)\right\}\\
&\le\mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k+1}(\xi_i+\eta_i)>t\right)\cap\left(\frac{1}{t}\sum_{i=1}^{k}\xi_i\le x\right)\right\}\\
&\le\mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(tx+\xi_{k+1}+\sum_{i=1}^{k+1}\eta_i>t\right)\cap\left(\frac{1}{t}\sum_{i=1}^{k}\xi_i\le x\right)\right\}\\
&=\mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\frac{\xi_{k+1}}{t}+\frac{1}{t}\sum_{i=1}^{k+1}\eta_i>1-x\right)\cap\left(\frac{1}{t}\sum_{i=1}^{k}\xi_i\le x\right)\right\}.
\end{aligned}$$

Since

$$\frac{\xi_{k+1}}{t}\to 0,\quad \text{as } t\to\infty$$

and

$$\sum_{i=1}^{k+1}\eta_i\sim(k+1)\eta_1,\quad \sum_{i=1}^{k}\xi_i\sim k\xi_1,$$

we have

$$\begin{aligned}
&\lim_{t\to\infty}\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\xi_i\le x\right\}\\
&\le\lim_{t\to\infty}\mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\eta_1>\frac{t(1-x)}{k+1}\right)\cap\left(\xi_1\le\frac{tx}{k}\right)\right\}\\
&=\lim_{t\to\infty}\sup_{k\ge 0}\mathcal{M}\left\{\eta_1>\frac{t(1-x)}{k+1}\right\}\wedge\mathcal{M}\left\{\xi_1\le\frac{tx}{k}\right\}\\
&=\lim_{t\to\infty}\sup_{k\ge 0}\left(1-\Psi\left(\frac{t(1-x)}{k+1}\right)\right)\wedge\Phi\left(\frac{tx}{k}\right)\\
&=\sup_{y>0}\Phi(xy)\wedge(1-\Psi(y-xy))=\Upsilon(x).
\end{aligned}$$

That is,

$$\lim_{t\to\infty}\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\xi_i\le x\right\}\le\Upsilon(x). \tag{13.44}$$

On the other hand, we have

$$
\begin{aligned}
&\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\xi_i > x\right\}\\
&= \mathcal{M}\left\{\bigcup_{k=0}^{\infty}(N_t = k)\cap\left(\frac{1}{t}\sum_{i=1}^{k+1}\xi_i > x\right)\right\}\\
&\le \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\sum_{i=1}^{k}(\xi_i+\eta_i)\le t\right)\cap\left(\frac{1}{t}\sum_{i=1}^{k+1}\xi_i > x\right)\right\}\\
&\le \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(tx-\xi_{k+1}+\sum_{i=1}^{k}\eta_i\le t\right)\cap\left(\frac{1}{t}\sum_{i=1}^{k+1}\xi_i > x\right)\right\}\\
&= \mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\frac{1}{t}\sum_{i=1}^{k}\eta_i-\frac{\xi_{k+1}}{t}\le 1-x\right)\cap\left(\frac{1}{t}\sum_{i=1}^{k+1}\xi_i > x\right)\right\}.
\end{aligned}
$$

Since

$$\frac{\xi_{k+1}}{t}\to 0, \quad \text{as } t\to\infty$$

and

$$\sum_{i=1}^{k}\eta_i\sim k\eta_1, \quad \sum_{i=1}^{k+1}\xi_i\sim (k+1)\xi_1,$$

we have

$$
\begin{aligned}
&\lim_{t\to\infty}\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\xi_i > x\right\}\\
&\le \lim_{t\to\infty}\mathcal{M}\left\{\bigcup_{k=0}^{\infty}\left(\eta_1\le\frac{t(1-x)}{k}\right)\cap\left(\xi_1>\frac{tx}{k+1}\right)\right\}\\
&= \lim_{t\to\infty}\sup_{k\ge 0}\mathcal{M}\left\{\eta_1\le\frac{t(1-x)}{k}\right\}\wedge\mathcal{M}\left\{\xi_1>\frac{tx}{k+1}\right\}\\
&= \lim_{t\to\infty}\sup_{k\ge 0}\Psi\left(\frac{t(1-x)}{k+1}\right)\wedge\left(1-\Phi\left(\frac{tx}{k+1}\right)\right)\\
&= \sup_{y>0}(1-\Phi(xy))\wedge\Psi(y-xy).
\end{aligned}
$$

By using the duality of uncertain measure, we get

$$
\begin{aligned}
\lim_{t\to\infty}\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\xi_i\le x\right\} &\ge 1-\sup_{y>0}(1-\Phi(xy))\wedge\Psi(y-xy)\\
&= \inf_{y>0}\Phi(xy)\vee(1-\Psi(y-xy)) = \Upsilon(x).
\end{aligned}
$$

That is,

$$\lim_{t\to\infty} \mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\xi_i \le x\right\} \ge \Upsilon(x). \tag{13.45}$$

Since

$$\frac{1}{t}\sum_{i=1}^{N_t}\xi_i \le \frac{A_t}{t} \le \frac{1}{t}\sum_{i=1}^{N_t+1}\xi_i,$$

we obtain

$$\mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t}\xi_i \le x\right\} \ge \mathcal{M}\left\{\frac{A_t}{t} \le x\right\} \ge \mathcal{M}\left\{\frac{1}{t}\sum_{i=1}^{N_t+1}\xi_i \le x\right\}.$$

It follows from (13.44) and (13.45) that for any real number x, we have

$$\lim_{t\to\infty}\left\{\frac{A_t}{t} \le x\right\} = \Upsilon(x).$$

Hence the availability rate A_t/t converges in distribution to $\xi_1/(\xi_1+\eta_1)$. The theorem is proved.

Theorem 13.14 *(Yao and Li [242], Alternating Renewal Theorem) Assume A_t is an alternating renewal process with uncertain on-times $\xi_1, \xi_2, \cdots$ and uncertain off-times $\eta_1, \eta_2, \cdots$ If $E[\xi_1/(\xi_1+\eta_1)]$ exists, then*

$$\lim_{t\to\infty}\frac{E[A_t]}{t} = E\left[\frac{\xi_1}{\xi_1+\eta_1}\right]. \tag{13.46}$$

If those on-times and off-times have regular uncertainty distributions Φ and Ψ, respectively, then

$$\lim_{t\to\infty}\frac{E[A_t]}{t} = \int_0^1 \frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha)+\Psi^{-1}(1-\alpha)}\mathrm{d}\alpha. \tag{13.47}$$

Proof: Write the uncertainty distributions of A_t/t and $\xi_1/(\xi_1+\eta_1)$ by $F_t(x)$ and $G(x)$, respectively. Since A_t/t converges in distribution to $\xi_1/(\xi_1+\eta_1)$, we have $F_t(x) \to G(x)$ as $t \to \infty$. It follows from Lebesgue dominated convergence theorem that

$$\lim_{t\to\infty}\frac{E[A_t]}{t} = \lim_{t\to\infty}\int_0^1 (1-F_t(x))\mathrm{d}x = \int_0^1 (1-G(x))\mathrm{d}x = E\left[\frac{\xi_1}{\xi_1+\eta_1}\right].$$

Finally, since the uncertain variable $\xi_1/(\xi_1+\eta_1)$ is strictly increasing with respect to ξ_1 and strictly decreasing with respect to η_1, it has an inverse uncertainty distribution

$$G^{-1}(\alpha) = \frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(\alpha)+\Psi(1-\alpha)}.$$

The equation (13.47) is thus obtained.

13.7 Bibliographic Notes

The concept of uncertain renewal process was first proposed by Liu [123] in 2008. Two years later, Liu [129] proved an uncertain elementary renewal theorem for determining the average renewal number. Liu [129] also provided the concept of uncertain renewal reward process and verified an uncertain renewal reward theorem for determining the long-run reward rate. In addition, Yao and Li [242] presented the concept of uncertain alternating renewal process and proved an uncertain alternating renewal theorem for determining the availability rate.

Based on the theory of uncertain renewal process, Liu [135] presented an uncertain insurance model by assuming the claim is an uncertain renewal reward process, and proved a formula for calculating ruin index. In addition, Yao [257] derived an uncertainty distribution of ruin time. Furthermore, Yao [250] discussed the uncertain block replacement policy, and Yao and Ralescu [245] investigated the uncertain age replacement policy and obtained the long-run average replacement cost.

Chapter 14

Uncertain Calculus

Uncertain calculus is a branch of mathematics that deals with differentiation and integration of uncertain processes. This chapter will introduce Liu process, Liu integral, fundamental theorem, chain rule, change of variables, and integration by parts.

14.1 Liu Process

In 2009, Liu [125] investigated a type of stationary independent increment process whose increments are normal uncertain variables. Later, this process was named by the academic community as Liu process due to its importance and usefulness. A formal definition is given below.

Definition 14.1 *(Liu [125]) An uncertain process C_t is said to be a canonical Liu process if*
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2.

It is clear that a canonical Liu process C_t is a stationary independent increment process and has a normal uncertainty distribution with expected value 0 and variance t^2. The uncertainty distribution of C_t is

$$\Phi_t(x) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}t}\right)\right)^{-1} \tag{14.1}$$

and inverse uncertainty distribution is

$$\Phi_t^{-1}(\alpha) = \frac{t\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} \tag{14.2}$$

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5_15

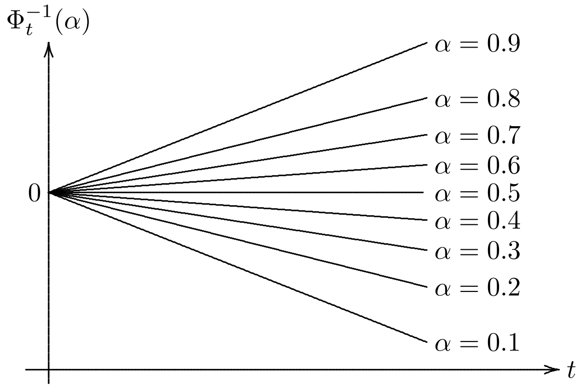


Figure 14.1: Inverse Uncertainty Distribution of Canonical Liu Process

that are homogeneous linear functions of time t for any given α. See Figure 14.1.

A canonical Liu process is defined by three properties in the above definition. Does such an uncertain process exist? The following theorem answers this question.

Theorem 14.1 *(Liu [129], Existence Theorem) There exists a canonical Liu process.*

Proof: It follows from Theorem 12.11 that there exists a stationary independent increment process C_t whose inverse uncertainty distribution is

$$\Phi_t^{-1}(\alpha) = \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha} t.$$

Furthermore, C_t has a Lipschitz continuous version. It is also easy to verify that every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2. Hence there exists a canonical Liu process.

Theorem 14.2 *Let C_t be a canonical Liu process. Then for each time $t >$ 0, the ratio C_t/t is a normal uncertain variable with expected value 0 and variance 1. That is,*

$$\frac{C_t}{t} \sim \mathcal{N}(0,1) \tag{14.3}$$

for any $t > 0$.

Proof: Since C_t is a normal uncertain variable $\mathcal{N}(0,t)$, the operational law tells us that C_t/t has an uncertainty distribution

$$\Psi(x) = \Phi_t(tx) = \left(1 + \exp\left(-\frac{\pi x}{\sqrt{3}}\right)\right)^{-1}.$$

Hence C_t/t is a normal uncertain variable with expected value 0 and variance 1. The theorem is verified.

Theorem 14.3 *(Liu [129]) Let C_t be a canonical Liu process. Then for each time t, we have*

$$\frac{t^2}{2} \le E[C_t^2] \le t^2. \tag{14.4}$$

Proof: Note that C_t is a normal uncertain variable and has an uncertainty distribution $\Phi_t(x)$ in (14.1). It follows from the definition of expected value that

$$E[C_t^2] = \int_0^{+\infty} \mathcal{M}\{C_t^2 \ge x\}\mathrm{d}x = \int_0^{+\infty} \mathcal{M}\{(C_t \ge \sqrt{x}) \cup (C_t \le -\sqrt{x})\}\mathrm{d}x.$$

On the one hand, we have

$$\begin{aligned} E[C_t^2] &\le \int_0^{+\infty} (\mathcal{M}\{C_t \ge \sqrt{x}\} + \mathcal{M}\{C_t \le -\sqrt{x}\})\mathrm{d}x \\ &= \int_0^{+\infty} (1 - \Phi_t(\sqrt{x}) + \Phi_t(-\sqrt{x}))\mathrm{d}x = t^2. \end{aligned}$$

On the other hand, we have

$$E[C_t^2] \ge \int_0^{+\infty} \mathcal{M}\{C_t \ge \sqrt{x}\}\mathrm{d}x = \int_0^{+\infty} (1 - \Phi_t(\sqrt{x}))\mathrm{d}x = \frac{t^2}{2}.$$

Hence (14.4) is proved.

Theorem 14.4 *(Iwamura and Xu [69]) Let C_t be a canonical Liu process. Then for each time t, we have*

$$1.24t^4 < V[C_t^2] < 4.31t^4. \tag{14.5}$$

Proof: Let q be the expected value of C_t^2. On the one hand, it follows from the definition of variance that

$$\begin{aligned} V[C_t^2] &= \int_0^{+\infty} \mathcal{M}\{(C_t^2 - q)^2 \ge x\}\mathrm{d}x \\ &\le \int_0^{+\infty} \mathcal{M}\left\{C_t \ge \sqrt{q + \sqrt{x}}\right\}\mathrm{d}x \\ &\quad + \int_0^{+\infty} \mathcal{M}\left\{C_t \le -\sqrt{q + \sqrt{x}}\right\}\mathrm{d}x \\ &\quad + \int_0^{+\infty} \mathcal{M}\left\{-\sqrt{q - \sqrt{x}} \le C_t \le \sqrt{q - \sqrt{x}}\right\}\mathrm{d}x. \end{aligned}$$

Since $t^2/2 \le q \le t^2$, we have

$$\begin{aligned}
\text{First Term } &= \int_0^{+\infty} \mathcal{M}\left\{C_t \ge \sqrt{q+\sqrt{x}}\right\}\mathrm{d}x \\
&\le \int_0^{+\infty} \mathcal{M}\left\{C_t \ge \sqrt{t^2/2+\sqrt{x}}\right\}\mathrm{d}x \\
&= \int_0^{+\infty} \left(1-\left(1+\exp\left(-\frac{\pi\sqrt{t^2/2+\sqrt{x}}}{\sqrt{3}t}\right)\right)^{-1}\right)\mathrm{d}x \\
&\le 1.725t^4,
\end{aligned}$$

$$\begin{aligned}
\text{Second Term } &= \int_0^{+\infty} \mathcal{M}\left\{C_t \le -\sqrt{q+\sqrt{x}}\right\}\mathrm{d}x \\
&\le \int_0^{+\infty} \mathcal{M}\left\{C_t \le -\sqrt{t^2/2+\sqrt{x}}\right\}\mathrm{d}x \\
&= \int_0^{+\infty} \left(1+\exp\left(\frac{\pi\sqrt{t^2/2+\sqrt{x}}}{\sqrt{3}t}\right)\right)^{-1}\mathrm{d}x \\
&\le 1.725t^4,
\end{aligned}$$

$$\begin{aligned}
\text{Third Term} &= \int_0^{+\infty} \mathcal{M}\left\{-\sqrt{q-\sqrt{x}} \le C_t \le \sqrt{q-\sqrt{x}}\right\}\mathrm{d}x \\
&\le \int_0^{+\infty} \mathcal{M}\left\{C_t \le \sqrt{q-\sqrt{x}}\right\}\mathrm{d}x \\
&\le \int_0^{+\infty} \mathcal{M}\left\{C_t \le \sqrt{t^2-\sqrt{x}}\right\}\mathrm{d}x \\
&= \int_0^{+\infty} \left(1+\exp\left(-\frac{\pi\sqrt{t^2+\sqrt{x}}}{\sqrt{3}t}\right)\right)^{-1}\mathrm{d}x \\
&< 0.86t^4.
\end{aligned}$$

It follows from the above three upper bounds that

$$V[C_t^2] < 1.725t^4 + 1.725t^4 + 0.86t^4 = 4.31t^4.$$

On the other hand, we have

$$\begin{aligned}
V[C_t^2] &= \int_0^{+\infty} \mathcal{M}\{(C_t^2 - q)^2 \ge x\}\mathrm{d}x \\
&\ge \int_0^{+\infty} \mathcal{M}\left\{C_t \ge \sqrt{q + \sqrt{x}}\right\}\mathrm{d}x \\
&\ge \int_0^{+\infty} \mathcal{M}\left\{C_t \ge \sqrt{t^2 + \sqrt{x}}\right\}\mathrm{d}x \\
&= \int_0^{+\infty} \left(1 - \left(1 + \exp\left(-\frac{\pi\sqrt{t^2+\sqrt{x}}}{\sqrt{3}t}\right)\right)^{-1}\right)\mathrm{d}x \\
&> 1.24t^4.
\end{aligned}$$

The theorem is thus verified. An open problem is to improve the bounds of the variance of the square of canonical Liu process.

Definition 14.2 *Let C_t be a canonical Liu process. Then for any real numbers e and $\sigma > 0$, the uncertain process*

$$A_t = et + \sigma C_t \tag{14.6}$$

is called an arithmetic Liu process, where e is called the drift and σ is called the diffusion.

It is clear that the arithmetic Liu process A_t is a type of stationary independent increment process. In addition, the arithmetic Liu process A_t has a normal uncertainty distribution with expected value et and variance $\sigma^2 t^2$, i.e.,

$$A_t \sim \mathcal{N}(et, \sigma t) \tag{14.7}$$

whose uncertainty distribution is

$$\Phi_t(x) = \left(1 + \exp\left(\frac{\pi(et - x)}{\sqrt{3}\sigma t}\right)\right)^{-1} \tag{14.8}$$

and inverse uncertainty distribution is

$$\Phi_t^{-1}(\alpha) = et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}. \tag{14.9}$$

Definition 14.3 *Let C_t be a canonical Liu process. Then for any real numbers e and $\sigma > 0$, the uncertain process*

$$G_t = \exp(et + \sigma C_t) \tag{14.10}$$

is called a geometric Liu process, where e is called the log-drift and σ is called the log-diffusion.

Note that the geometric Liu process G_t has a lognormal uncertainty distribution, i.e.,

$$G_t \sim \mathcal{LOGN}(et, \sigma t) \tag{14.11}$$

whose uncertainty distribution is

$$\Phi_t(x) = \left(1 + \exp\left(\frac{\pi(et - \ln x)}{\sqrt{3}\sigma t}\right)\right)^{-1} \tag{14.12}$$

and inverse uncertainty distribution is

$$\Phi_t^{-1}(\alpha) = \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right). \tag{14.13}$$

Furthermore, the geometric Liu process G_t has an expected value,

$$E[G_t] = \begin{cases} \sigma t\sqrt{3}\exp(et)\csc(\sigma t\sqrt{3}), & \text{if } t < \pi/(\sigma\sqrt{3}) \\ +\infty, & \text{if } t \ge \pi/(\sigma\sqrt{3}). \end{cases} \tag{14.14}$$

14.2 Liu Integral

As the most popular topic of uncertain integral, Liu integral allows us to integrate an uncertain process (the integrand) with respect to Liu process (the integrator). The result of Liu integral is another uncertain process.

Definition 14.4 *(Liu [125]) Let X_t be an uncertain process and let C_t be a canonical Liu process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1\le i\le k} |t_{i+1} - t_i|. \tag{14.15}$$

Then Liu integral of X_t with respect to C_t is defined as

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta\to 0}\sum_{i=1}^{k} X_{t_i}\cdot(C_{t_{i+1}} - C_{t_i}) \tag{14.16}$$

provided that the limit exists almost surely and is finite. In this case, the uncertain process X_t is said to be integrable.

Since X_t and C_t are uncertain variables at each time t, the limit in (14.16) is also an uncertain variable provided that the limit exists almost surely and is finite. Hence an uncertain process X_t is integrable with respect to C_t if and only if the limit in (14.16) is an uncertain variable.

Example 14.1: For any partition $0 = t_1 < t_2 < \cdots < t_{k+1} = s$, it follows from (14.16) that

$$\int_0^s \mathrm{d}C_t = \lim_{\Delta\to 0}\sum_{i=1}^{k}(C_{t_{i+1}} - C_{t_i}) \equiv C_s - C_0 = C_s.$$

That is,

$$\int_0^s \mathrm{d}C_t = C_s. \tag{14.17}$$

Example 14.2: For any partition $0 = t_1 < t_2 < \cdots < t_{k+1} = s$, it follows from (14.16) that

$$\begin{aligned} C_s^2 &= \sum_{i=1}^{k}\left(C_{t_{i+1}}^2 - C_{t_i}^2\right) \\ &= \sum_{i=1}^{k}\left(C_{t_{i+1}} - C_{t_i}\right)^2 + 2\sum_{i=1}^{k} C_{t_i}\left(C_{t_{i+1}} - C_{t_i}\right) \\ &\to 0 + 2\int_0^s C_t \mathrm{d}C_t \end{aligned}$$

as $\Delta \to 0$. That is,

$$\int_0^s C_t \mathrm{d}C_t = \frac{1}{2}C_s^2. \tag{14.18}$$

Example 14.3: For any partition $0 = t_1 < t_2 < \cdots < t_{k+1} = s$, it follows from (14.16) that

$$\begin{aligned} sC_s &= \sum_{i=1}^{k}\left(t_{i+1}C_{t_{i+1}} - t_i C_{t_i}\right) \\ &= \sum_{i=1}^{k} C_{t_{i+1}}(t_{i+1} - t_i) + \sum_{i=1}^{k} t_i(C_{t_{i+1}} - C_{t_i}) \\ &\to \int_0^s C_t \mathrm{d}t + \int_0^s t \mathrm{d}C_t \end{aligned}$$

as $\Delta \to 0$. That is,

$$\int_0^s C_t \mathrm{d}t + \int_0^s t \mathrm{d}C_t = sC_s. \tag{14.19}$$

Theorem 14.5 *If X_t is a sample-continuous uncertain process on $[a, b]$, then it is integrable with respect to C_t on $[a, b]$.*

Proof: Let $a = t_1 < t_2 < \cdots < t_{k+1} = b$ be a partition of the closed interval $[a, b]$. Since the uncertain process X_t is sample-continuous, almost all sample paths are continuous functions with respect to t. Hence the limit

$$\lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i})$$

exists almost surely and is finite. On the other hand, since X_t and C_t are uncertain variables at each time t, the above limit is also a measurable function. Hence the limit is an uncertain variable and then X_t is integrable with respect to C_t.

Theorem 14.6 *If X_t is an integrable uncertain process on $[a, b]$, then it is integrable on each subinterval of $[a, b]$. Moreover, if $c \in [a, b]$, then*

$$\int_a^b X_t \mathrm{d}C_t = \int_a^c X_t \mathrm{d}C_t + \int_c^b X_t \mathrm{d}C_t. \tag{14.20}$$

Proof: Let $[a', b']$ be a subinterval of $[a, b]$. Since X_t is an integrable uncertain process on $[a, b]$, for any partition

$$a = t_1 < \cdots < t_m = a' < t_{m+1} < \cdots < t_n = b' < t_{n+1} < \cdots < t_{k+1} = b,$$

the limit

$$\lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i})$$

exists almost surely and is finite. Thus the limit

$$\lim_{\Delta \to 0} \sum_{i=m}^{n-1} X_{t_i}(C_{t_{i+1}} - C_{t_i})$$

exists almost surely and is finite. Hence X_t is integrable on the subinterval $[a', b']$. Next, for the partition

$$a = t_1 < \cdots < t_m = c < t_{m+1} < \cdots < t_{k+1} = b,$$

we have

$$\sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}) = \sum_{i=1}^{m-1} X_{t_i}(C_{t_{i+1}} - C_{t_i}) + \sum_{i=m}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}).$$

Note that

$$\int_a^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}),$$

$$\int_a^c X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{m-1} X_{t_i}(C_{t_{i+1}} - C_{t_i}),$$

$$\int_c^b X_t \mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=m}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}).$$

Hence the equation (14.20) is proved.

Theorem 14.7 *(Linearity of Liu Integral) Let X_t and Y_t be integrable uncertain processes on $[a,b]$, and let α and β be real numbers. Then*

$$\int_a^b (\alpha X_t + \beta Y_t)\mathrm{d}C_t = \alpha \int_a^b X_t \mathrm{d}C_t + \beta \int_a^b Y_t \mathrm{d}C_t. \tag{14.21}$$

Proof: Let $a = t_1 < t_2 < \cdots < t_{k+1} = b$ be a partition of the closed interval $[a,b]$. It follows from the definition of Liu integral that

$$\int_a^b (\alpha X_t + \beta Y_t)\mathrm{d}C_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} (\alpha X_{t_i} + \beta Y_{t_i})(C_{t_{i+1}} - C_{t_i})$$

$$= \lim_{\Delta \to 0} \alpha \sum_{i=1}^{k} X_{t_i}(C_{t_{i+1}} - C_{t_i}) + \lim_{\Delta \to 0} \beta \sum_{i=1}^{k} Y_{t_i}(C_{t_{i+1}} - C_{t_i})$$

$$= \alpha \int_a^b X_t \mathrm{d}C_t + \beta \int_a^b Y_t \mathrm{d}C_t.$$

Hence the equation (14.21) is proved.

Theorem 14.8 *Let $f(t)$ be an integrable function with respect to t. Then the Liu integral*

$$\int_0^s f(t)\mathrm{d}C_t \tag{14.22}$$

is a normal uncertain variable at each time s, and

$$\int_0^s f(t)\mathrm{d}C_t \sim \mathcal{N}\left(0, \int_0^s |f(t)|\mathrm{d}t\right). \tag{14.23}$$

Proof: Since the increments of C_t are stationary and independent normal uncertain variables, for any partition of closed interval $[0,s]$ with $0 = t_1 < t_2 < \cdots < t_{k+1} = s$, it follows from Theorem 2.12 that

$$\sum_{i=1}^{k} f(t_i)(C_{t_{i+1}} - C_{t_i}) \sim \mathcal{N}\left(0, \sum_{i=1}^{k} |f(t_i)|(t_{i+1} - t_i)\right).$$

That is, the sum is also a normal uncertain variable. Since f is an integrable function, we have

$$\sum_{i=1}^{k} |f(t_i)|(t_{i+1} - t_i) \to \int_0^s |f(t)|\mathrm{d}t$$

as the mesh $\Delta \to 0$. Hence we obtain

$$\int_0^s f(t)\mathrm{d}C_t = \lim_{\Delta\to 0}\sum_{i=1}^k f(t_i)(C_{t_{i+1}} - C_{t_i}) \sim \mathcal{N}\left(0, \int_0^s |f(t)|\mathrm{d}t\right).$$

The theorem is proved.

Exercise 14.1: Let s be a given time with $s > 0$. Show that the Liu integral

$$\int_0^s t\mathrm{d}C_t \tag{14.24}$$

is a normal uncertain variable $\mathcal{N}(0, s^2/2)$ and has an uncertainty distribution

$$\Phi_s(x) = \left(1 + \exp\left(-\frac{2\pi x}{\sqrt{3}s^2}\right)\right)^{-1}. \tag{14.25}$$

Exercise 14.2: For any real number α with $0 < \alpha < 1$, the uncertain process

$$F_s = \int_0^s (s-t)^{-\alpha}\mathrm{d}C_t \tag{14.26}$$

is called a *fractional Liu process* with index α. Show that F_s is a normal uncertain variable and

$$F_s \sim \mathcal{N}\left(0, \frac{s^{1-\alpha}}{1-\alpha}\right) \tag{14.27}$$

whose uncertainty distribution is

$$\Phi_s(x) = \left(1 + \exp\left(-\frac{\pi(1-\alpha)x}{\sqrt{3}s^{1-\alpha}}\right)\right)^{-1}. \tag{14.28}$$

Definition 14.5 *(Chen and Ralescu [20]) Let C_t be a canonical Liu process and let Z_t be an uncertain process. If there exist uncertain processes μ_t and σ_t such that*

$$Z_t = Z_0 + \int_0^t \mu_s\mathrm{d}s + \int_0^t \sigma_s\mathrm{d}C_s \tag{14.29}$$

for any $t \ge 0$, then Z_t is called a Liu process with drift μ_t and diffusion σ_t. Furthermore, Z_t has an uncertain differential

$$\mathrm{d}Z_t = \mu_t\mathrm{d}t + \sigma_t\mathrm{d}C_t. \tag{14.30}$$

Example 14.4: It follows from the equation (14.17) that the canonical Liu process C_t can be written as

$$C_t = \int_0^t \mathrm{d}C_s.$$

Thus C_t is a Liu process with drift 0 and diffusion 1, and has an uncertain differential $\mathrm{d}C_t$.

Example 14.5: It follows from the equation (14.18) that C_t^2 can be written as

$$C_t^2 = 2\int_0^t C_s \mathrm{d}C_s.$$

Thus C_t^2 is a Liu process with drift 0 and diffusion $2C_t$, and has an uncertain differential

$$\mathrm{d}(C_t^2) = 2C_t\mathrm{d}C_t.$$

Example 14.6: It follows from the equation (14.19) that tC_t can be written as

$$tC_t = \int_0^t C_s\mathrm{d}s + \int_0^t s\mathrm{d}C_s.$$

Thus tC_t is a Liu process with drift C_t and diffusion t, and has an uncertain differential

$$\mathrm{d}(tC_t) = C_t\mathrm{d}t + t\mathrm{d}C_t.$$

Theorem 14.9 *(Chen and Ralescu [20]) Liu process is a sample-continuous uncertain process.*

Proof: Let Z_t be a Liu process. Then there exist two uncertain processes μ_t and σ_t such that

$$Z_t = Z_0 + \int_0^t \mu_s\mathrm{d}s + \int_0^t \sigma_s\mathrm{d}C_s.$$

For each $\gamma \in \Gamma$, we have

$$|Z_t(\gamma) - Z_r(\gamma)| = \left|\int_r^t \mu_s(\gamma)\mathrm{d}s + \int_r^t \sigma_s(\gamma)\mathrm{d}C_s(\gamma)\right| \to 0$$

as $r \to t$. Thus Z_t is sample-continuous and the theorem is proved.

14.3 Fundamental Theorem

Theorem 14.10 *(Liu [125], Fundamental Theorem of Uncertain Calculus) Let $h(t,c)$ be a continuously differentiable function. Then $Z_t = h(t, C_t)$ is a Liu process and has an uncertain differential*

$$\mathrm{d}Z_t = \frac{\partial h}{\partial t}(t, C_t)\mathrm{d}t + \frac{\partial h}{\partial c}(t, C_t)\mathrm{d}C_t. \tag{14.31}$$

Proof: Write $\Delta C_t = C_{t+\Delta t} - C_t = C_{\Delta t}$. It follows from Theorems 14.3 and 14.4 that Δt and ΔC_t are infinitesimals with the same order. Since the function h is continuously differentiable, by using Taylor series expansion, the infinitesimal increment of Z_t has a first-order approximation,

$$\Delta Z_t = \frac{\partial h}{\partial t}(t, C_t)\Delta t + \frac{\partial h}{\partial c}(t, C_t)\Delta C_t.$$

Hence we obtain the uncertain differential (14.31) because it makes

$$Z_s = Z_0 + \int_0^s \frac{\partial h}{\partial t}(t, C_t)\mathrm{d}t + \int_0^s \frac{\partial h}{\partial c}(t, C_t)\mathrm{d}C_t. \tag{14.32}$$

This formula is an integral form of the fundamental theorem.

Example 14.7: Let us calculate the uncertain differential of tC_t. In this case, we have $h(t, c) = tc$ whose partial derivatives are

$$\frac{\partial h}{\partial t}(t, c) = c, \quad \frac{\partial h}{\partial c}(t, c) = t.$$

It follows from the fundamental theorem of uncertain calculus that

$$\mathrm{d}(tC_t) = C_t\mathrm{d}t + t\mathrm{d}C_t. \tag{14.33}$$

Thus tC_t is a Liu process with drift C_t and diffusion t.

Example 14.8: Let us calculate the uncertain differential of the arithmetic Liu process $A_t = et + \sigma C_t$. In this case, we have $h(t, c) = et + \sigma c$ whose partial derivatives are

$$\frac{\partial h}{\partial t}(t, c) = e, \quad \frac{\partial h}{\partial c}(t, c) = \sigma.$$

It follows from the fundamental theorem of uncertain calculus that

$$\mathrm{d}A_t = e\mathrm{d}t + \sigma\mathrm{d}C_t. \tag{14.34}$$

Thus A_t is a Liu process with drift e and diffusion σ.

Example 14.9: Let us calculate the uncertain differential of the geometric Liu process $G_t = \exp(et + \sigma C_t)$. In this case, we have $h(t, c) = \exp(et + \sigma c)$ whose partial derivatives are

$$\frac{\partial h}{\partial t}(t, c) = eh(t, c), \quad \frac{\partial h}{\partial c}(t, c) = \sigma h(t, c).$$

It follows from the fundamental theorem of uncertain calculus that

$$\mathrm{d}G_t = eG_t\mathrm{d}t + \sigma G_t\mathrm{d}C_t. \tag{14.35}$$

Thus G_t is a Liu process with drift eG_t and diffusion σG_t.

14.4 Chain Rule

Chain rule is a special case of the fundamental theorem of uncertain calculus.

Theorem 14.11 *(Liu [125], Chain Rule) Let $f(c)$ be a continuously differentiable function. Then $f(C_t)$ has an uncertain differential*

$$\mathrm{d}f(C_t) = f'(C_t)\mathrm{d}C_t. \tag{14.36}$$

Proof: Since $f(c)$ is a continuously differentiable function, we immediately have

$$\frac{\partial}{\partial t}f(c) = 0, \quad \frac{\partial}{\partial c}f(c) = f'(c).$$

It follows from the fundamental theorem of uncertain calculus that the equation (14.36) holds.

Example 14.10: Let us calculate the uncertain differential of C_t^2. In this case, we have $f(c) = c^2$ and $f'(c) = 2c$. It follows from the chain rule that

$$\mathrm{d}C_t^2 = 2C_t\mathrm{d}C_t. \tag{14.37}$$

Example 14.11: Let us calculate the uncertain differential of $\sin(C_t)$. In this case, we have $f(c) = \sin(c)$ and $f'(c) = \cos(c)$. It follows from the chain rule that

$$\mathrm{d}\sin(C_t) = \cos(C_t)\mathrm{d}C_t. \tag{14.38}$$

Example 14.12: Let us calculate the uncertain differential of $\exp(C_t)$. In this case, we have $f(c) = \exp(c)$ and $f'(c) = \exp(c)$. It follows from the chain rule that

$$\mathrm{d}\exp(C_t) = \exp(C_t)\mathrm{d}C_t. \tag{14.39}$$

14.5 Change of Variables

Theorem 14.12 *(Liu [125], Change of Variables) Let f be a continuously differentiable function. Then for any $s > 0$, we have*

$$\int_0^s f'(C_t)\mathrm{d}C_t = \int_{C_0}^{C_s} f'(c)\mathrm{d}c. \tag{14.40}$$

That is,

$$\int_0^s f'(C_t)\mathrm{d}C_t = f(C_s) - f(C_0). \tag{14.41}$$

Proof: Since f is a continuously differentiable function, it follows from the chain rule that

$$\mathrm{d}f(C_t) = f'(C_t)\mathrm{d}C_t.$$

By using the fundamental theorem of uncertain calculus, we get

$$f(C_s) = f(C_0) + \int_0^s f'(C_t)\mathrm{d}C_t.$$

Hence the theorem is verified.

Example 14.13: Since the function $f(c) = c$ has an antiderivative $c^2/2$, it follows from the change of variables of integral that

$$\int_0^s C_t\mathrm{d}C_t = \frac{1}{2}C_s^2 - \frac{1}{2}C_0^2 = \frac{1}{2}C_s^2.$$

Example 14.14: Since the function $f(c) = c^2$ has an antiderivative $c^3/3$, it follows from the change of variables of integral that

$$\int_0^s C_t^2\mathrm{d}C_t = \frac{1}{3}C_s^3 - \frac{1}{3}C_0^3 = \frac{1}{3}C_s^3.$$

Example 14.15: Since the function $f(c) = \exp(c)$ has an antiderivative $\exp(c)$, it follows from the change of variables of integral that

$$\int_0^s \exp(C_t)\mathrm{d}C_t = \exp(C_s) - \exp(C_0) = \exp(C_s) - 1.$$

14.6 Integration by Parts

Theorem 14.13 *(Liu [125], Integration by Parts) Suppose X_t and Y_t are Liu processes. Then*

$$\mathrm{d}(X_tY_t) = Y_t\mathrm{d}X_t + X_t\mathrm{d}Y_t. \tag{14.42}$$

Proof: Note that ΔX_t and ΔY_t are infinitesimals with the same order. Since the function xy is a continuously differentiable function with respect to x and y, by using Taylor series expansion, the infinitesimal increment of X_tY_t has a first-order approximation,

$$\Delta(X_tY_t) = Y_t\Delta X_t + X_t\Delta Y_t.$$

Hence we obtain the uncertain differential (14.42) because it makes

$$X_sY_s = X_0Y_0 + \int_0^s Y_t\mathrm{d}X_t + \int_0^s X_t\mathrm{d}Y_t. \tag{14.43}$$

The theorem is thus proved.

Example 14.16: In order to illustrate the integration by parts, let us calculate the uncertain differential of

$$Z_t = \exp(t)C_t^2.$$

In this case, we define

$$X_t = \exp(t), \quad Y_t = C_t^2.$$

Then

$$\mathrm{d}X_t = \exp(t)\mathrm{d}t, \quad \mathrm{d}Y_t = 2C_t\mathrm{d}C_t.$$

It follows from the integration by parts that

$$\mathrm{d}Z_t = \exp(t)C_t^2\mathrm{d}t + 2\exp(t)C_t\mathrm{d}C_t.$$

Example 14.17: The integration by parts may also calculate the uncertain differential of

$$Z_t = \sin(t+1)\int_0^t s\mathrm{d}C_s.$$

In this case, we define

$$X_t = \sin(t+1), \quad Y_t = \int_0^t s\mathrm{d}C_s.$$

Then

$$\mathrm{d}X_t = \cos(t+1)\mathrm{d}t, \quad \mathrm{d}Y_t = t\mathrm{d}C_t.$$

It follows from the integration by parts that

$$\mathrm{d}Z_t = \left(\int_0^t s\mathrm{d}C_s\right)\cos(t+1)\mathrm{d}t + \sin(t+1)t\mathrm{d}C_t.$$

Example 14.18: Let f and g be continuously differentiable functions. It is clear that

$$Z_t = f(t)g(C_t)$$

is an uncertain process. In order to calculate the uncertain differential of Z_t, we define

$$X_t = f(t), \quad Y_t = g(C_t)$$

Then

$$\mathrm{d}X_t = f'(t)\mathrm{d}t, \quad \mathrm{d}Y_t = g'(C_t)\mathrm{d}C_t.$$

It follows from the integration by parts that

$$\mathrm{d}Z_t = f'(t)g(C_t)\mathrm{d}t + f(t)g'(C_t)\mathrm{d}C_t.$$

14.7 Bibliographic Notes

The concept of uncertain integral was first proposed by Liu [123] in 2008 in order to integrate uncertain processes with respect to Liu process. One year later, Liu [125] recast his work via the fundamental theorem of uncertain calculus from which the techniques of chain rule, change of variables, and integration by parts were derived.

Note that uncertain integral may also be defined with respect to other integrators. For example, Liu and Yao [132] suggested an uncertain integral with respect to multiple Liu processes. In addition, Chen and Ralescu [20] presented an uncertain integral with respect to general Liu process. In order to deal with uncertain process with jumps, Yao integral [241] was defined as a type of uncertain integral with respect to uncertain renewal process. Since then, the theory of uncertain calculus was well developed.

Chapter 15

Uncertain Differential Equation

Uncertain differential equation is a type of differential equation involving uncertain processes. This chapter will discuss the existence, uniqueness and stability of solutions of uncertain differential equations, and introduce Yao-Chen formula that represents the solution of an uncertain differential equation by a family of solutions of ordinary differential equations. On the basis of this formula, some formulas to calculate extreme value, first hitting time, and time integral of solution are provided. Furthermore, some numerical methods for solving general uncertain differential equations are designed.

15.1 Uncertain Differential Equation

Definition 15.1 *(Liu [123]) Suppose C_t is a canonical Liu process, and f and g are two functions. Then*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{15.1}$$

is called an uncertain differential equation. A solution is a Liu process X_t that satisfies (15.1) identically in t.

Remark 15.1: The uncertain differential equation (15.1) is equivalent to the uncertain integral equation

$$X_s = X_0 + \int_0^s f(t, X_t)\mathrm{d}t + \int_0^s g(t, X_t)\mathrm{d}C_t. \tag{15.2}$$

Theorem 15.1 *Let u_t and v_t be two integrable uncertain processes. Then the uncertain differential equation*

$$\mathrm{d}X_t = u_t\mathrm{d}t + v_t\mathrm{d}C_t \tag{15.3}$$

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5_16

has a solution

$$X_t = X_0 + \int_0^t u_s \mathrm{d}s + \int_0^t v_s \mathrm{d}C_s. \tag{15.4}$$

Proof: This theorem is essentially the definition of uncertain differential or a direct deduction of the fundamental theorem of uncertain calculus.

Example 15.1: Let a and b be real numbers. Consider the uncertain differential equation

$$\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t. \tag{15.5}$$

It follows from Theorem 15.1 that the solution is

$$X_t = X_0 + \int_0^t a\mathrm{d}s + \int_0^t b\mathrm{d}C_s.$$

That is,

$$X_t = X_0 + at + bC_t. \tag{15.6}$$

Theorem 15.2 *Let u_t and v_t be two integrable uncertain processes. Then the uncertain differential equation*

$$\mathrm{d}X_t = u_t X_t \mathrm{d}t + v_t X_t \mathrm{d}C_t \tag{15.7}$$

has a solution

$$X_t = X_0 \exp\left(\int_0^t u_s \mathrm{d}s + \int_0^t v_s \mathrm{d}C_s\right). \tag{15.8}$$

Proof: At first, the original uncertain differential equation is equivalent to

$$\frac{\mathrm{d}X_t}{X_t} = u_t \mathrm{d}t + v_t \mathrm{d}C_t.$$

It follows from the fundamental theorem of uncertain calculus that

$$\mathrm{d}\ln X_t = \frac{\mathrm{d}X_t}{X_t} = u_t \mathrm{d}t + v_t \mathrm{d}C_t$$

and then

$$\ln X_t = \ln X_0 + \int_0^t u_s \mathrm{d}s + \int_0^t v_s \mathrm{d}C_s.$$

Therefore the uncertain differential equation has a solution (15.8).

Example 15.2: Let a and b be real numbers. Consider the uncertain differential equation

$$\mathrm{d}X_t = aX_t \mathrm{d}t + bX_t \mathrm{d}C_t. \tag{15.9}$$

It follows from Theorem 15.2 that the solution is

$$X_t = X_0 \exp\left(\int_0^t a\mathrm{d}s + \int_0^t b\mathrm{d}C_s\right).$$

That is,

$$X_t = X_0 \exp\left(at + bC_t\right). \tag{15.10}$$

Linear Uncertain Differential Equation

Theorem 15.3 *(Chen and Liu [12]) Let $u_{1t}, u_{2t}, v_{1t}, v_{2t}$ be integrable uncertain processes. Then the linear uncertain differential equation*

$$\mathrm{d}X_t = (u_{1t}X_t + u_{2t})\mathrm{d}t + (v_{1t}X_t + v_{2t})\mathrm{d}C_t \tag{15.11}$$

has a solution

$$X_t = U_t\left(X_0 + \int_0^t \frac{u_{2s}}{U_s}\mathrm{d}s + \int_0^t \frac{v_{2s}}{U_s}\mathrm{d}C_s\right) \tag{15.12}$$

where

$$U_t = \exp\left(\int_0^t u_{1s}\mathrm{d}s + \int_0^t v_{1s}\mathrm{d}C_s\right). \tag{15.13}$$

Proof: At first, we define two uncertain processes U_t and V_t via uncertain differential equations,

$$\mathrm{d}U_t = u_{1t}U_t\mathrm{d}t + v_{1t}U_t\mathrm{d}C_t, \quad \mathrm{d}V_t = \frac{u_{2t}}{U_t}\mathrm{d}t + \frac{v_{2t}}{U_t}\mathrm{d}C_t.$$

It follows from the integration by parts that

$$\mathrm{d}(U_tV_t) = V_t\mathrm{d}U_t + U_t\mathrm{d}V_t = (u_{1t}U_tV_t + u_{2t})\mathrm{d}t + (v_{1t}U_tV_t + v_{2t})\mathrm{d}C_t.$$

That is, the uncertain process $X_t = U_tV_t$ is a solution of the uncertain differential equation (15.11). Note that

$$U_t = U_0\exp\left(\int_0^t u_{1s}\mathrm{d}s + \int_0^t v_{1s}\mathrm{d}C_s\right),$$

$$V_t = V_0 + \int_0^t \frac{u_{2s}}{U_s}\mathrm{d}s + \int_0^t \frac{v_{2s}}{U_s}\mathrm{d}C_s.$$

Taking $U_0 = 1$ and $V_0 = X_0$, we get the solution (15.12). The theorem is proved.

Example 15.3: Let m, a, σ be real numbers. Consider a linear uncertain differential equation

$$\mathrm{d}X_t = (m - aX_t)\mathrm{d}t + \sigma\mathrm{d}C_t. \tag{15.14}$$

At first, we have

$$U_t = \exp\left(\int_0^t (-a)\mathrm{d}s + \int_0^t 0\mathrm{d}C_s\right) = \exp(-at).$$

It follows from Theorem 15.3 that the solution is

$$X_t = \exp(-at)\left(X_0 + \int_0^t m\exp(as)\mathrm{d}s + \int_0^t \sigma\exp(as)\mathrm{d}C_s\right).$$

That is,

$$X_t = \frac{m}{a} + \exp(-at)\left(X_0 - \frac{m}{a}\right) + \sigma\exp(-at)\int_0^t \exp(as)\mathrm{d}C_s \qquad (15.15)$$

provided that $a \neq 0$. Note that X_t is a normal uncertain variable, i.e.,

$$X_t \sim \mathcal{N}\left(\frac{m}{a} + \exp(-at)\left(X_0 - \frac{m}{a}\right),\ \frac{\sigma}{a} - \exp(-at)\frac{\sigma}{a}\right). \qquad (15.16)$$

Example 15.4: Let m and σ be real numbers. Consider a linear uncertain differential equation

$$\mathrm{d}X_t = m\mathrm{d}t + \sigma X_t\mathrm{d}C_t. \qquad (15.17)$$

At first, we have

$$U_t = \exp\left(\int_0^t 0\mathrm{d}s + \int_0^t \sigma\mathrm{d}C_s\right) = \exp(\sigma C_t).$$

It follows from Theorem 15.3 that the solution is

$$X_t = \exp(\sigma C_t)\left(X_0 + \int_0^t m\exp(-\sigma s)\mathrm{d}s + \int_0^t 0\mathrm{d}C_s\right).$$

That is,

$$X_t = \exp(\sigma C_t)\left(X_0 + m\int_0^t \exp(-\sigma C_s)\mathrm{d}s\right). \qquad (15.18)$$

15.2 Analytic Methods

This section will provide two analytic methods for solving some nonlinear uncertain differential equations.

First Analytic Method

This subsection will introduce an analytic method for solving nonlinear uncertain differential equations like

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \sigma_t X_t\mathrm{d}C_t \qquad (15.19)$$

and

$$\mathrm{d}X_t = \alpha_t X_t\mathrm{d}t + g(t, X_t)\mathrm{d}C_t. \qquad (15.20)$$

Theorem 15.4 *(Liu [148]) Let f be a function of two variables and let σ_t be an integrable uncertain process. Then the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \sigma_t X_t\mathrm{d}C_t \qquad (15.21)$$

has a solution

$$X_t = Y_t^{-1} Z_t \tag{15.22}$$

where

$$Y_t = \exp\left(-\int_0^t \sigma_s \mathrm{d}C_s\right) \tag{15.23}$$

and Z_t is the solution of the uncertain differential equation

$$\mathrm{d}Z_t = Y_t f(t, Y_t^{-1} Z_t)\mathrm{d}t \tag{15.24}$$

with initial value $Z_0 = X_0$.

Proof: At first, by using the chain rule, the uncertain process Y_t has an uncertain differential

$$\mathrm{d}Y_t = -\exp\left(-\int_0^t \sigma_s \mathrm{d}C_s\right)\sigma_t \mathrm{d}C_t = -Y_t\sigma_t\mathrm{d}C_t.$$

It follows from the integration by parts that

$$\mathrm{d}(X_tY_t) = X_t\mathrm{d}Y_t + Y_t\mathrm{d}X_t = -X_tY_t\sigma_t\mathrm{d}C_t + Y_tf(t,X_t)\mathrm{d}t + Y_t\sigma_tX_t\mathrm{d}C_t.$$

That is,

$$\mathrm{d}(X_tY_t) = Y_tf(t,X_t)\mathrm{d}t.$$

Defining $Z_t = X_tY_t$, we obtain $X_t = Y_t^{-1}Z_t$ and $\mathrm{d}Z_t = Y_tf(t, Y_t^{-1}Z_t)\mathrm{d}t$. Furthermore, since $Y_0 = 1$, the initial value Z_0 is just X_0. The theorem is thus verified.

Example 15.5: Let α and σ be real numbers with $\alpha \neq 1$. Consider the uncertain differential equation

$$\mathrm{d}X_t = X_t^\alpha \mathrm{d}t + \sigma X_t \mathrm{d}C_t. \tag{15.25}$$

At first, we have $Y_t = \exp(-\sigma C_t)$ and Z_t satisfies the uncertain differential equation,

$$\mathrm{d}Z_t = \exp(-\sigma C_t)(\exp(\sigma C_t)Z_t)^\alpha \mathrm{d}t = \exp((\alpha-1)\sigma C_t)Z_t^\alpha \mathrm{d}t.$$

Since $\alpha \neq 1$, we have

$$\mathrm{d}Z_t^{1-\alpha} = (1-\alpha)\exp((\alpha-1)\sigma C_t)\mathrm{d}t.$$

It follows from the fundamental theorem of uncertain calculus that

$$Z_t^{1-\alpha} = Z_0^{1-\alpha} + (1-\alpha)\int_0^t \exp((\alpha-1)\sigma C_s)\mathrm{d}s.$$

Since the initial value Z_0 is just X_0, we have

$$Z_t = \left(X_0^{1-\alpha} + (1-\alpha)\int_0^t \exp((\alpha-1)\sigma C_s)\mathrm{d}s\right)^{1/(1-\alpha)}.$$

Theorem 15.4 says the uncertain differential equation (15.25) has a solution $X_t = Y_t^{-1}Z_t$, i.e.,

$$X_t = \exp(\sigma C_t)\left(X_0^{1-\alpha} + (1-\alpha)\int_0^t \exp((\alpha-1)\sigma C_s)\mathrm{d}s\right)^{1/(1-\alpha)}.$$

Theorem 15.5 *(Liu [148]) Let g be a function of two variables and let α_t be an integrable uncertain process. Then the uncertain differential equation*

$$\mathrm{d}X_t = \alpha_t X_t \mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{15.26}$$

has a solution

$$X_t = Y_t^{-1}Z_t \tag{15.27}$$

where

$$Y_t = \exp\left(-\int_0^t \alpha_s \mathrm{d}s\right) \tag{15.28}$$

and Z_t is the solution of the uncertain differential equation

$$\mathrm{d}Z_t = Y_t g(t, Y_t^{-1}Z_t)\mathrm{d}C_t \tag{15.29}$$

with initial value $Z_0 = X_0$.

Proof: At first, by using the chain rule, the uncertain process Y_t has an uncertain differential

$$\mathrm{d}Y_t = -\exp\left(-\int_0^t \alpha_s \mathrm{d}s\right)\alpha_t \mathrm{d}t = -Y_t\alpha_t\mathrm{d}t.$$

It follows from the integration by parts that

$$\mathrm{d}(X_tY_t) = X_t\mathrm{d}Y_t + Y_t\mathrm{d}X_t = -X_tY_t\alpha_t\mathrm{d}t + Y_t\alpha_tX_t\mathrm{d}t + Y_tg(t, X_t)\mathrm{d}C_t.$$

That is,

$$\mathrm{d}(X_tY_t) = Y_tg(t, X_t)\mathrm{d}C_t.$$

Defining $Z_t = X_tY_t$, we obtain $X_t = Y_t^{-1}Z_t$ and $\mathrm{d}Z_t = Y_tg(t, Y_t^{-1}Z_t)\mathrm{d}C_t$. Furthermore, since $Y_0 = 1$, the initial value Z_0 is just X_0. The theorem is thus verified.

Example 15.6: Let α and β be real numbers with $\beta \neq 1$. Consider the uncertain differential equation

$$\mathrm{d}X_t = \alpha X_t\mathrm{d}t + X_t^\beta \mathrm{d}C_t. \tag{15.30}$$

At first, we have $Y_t = \exp(-\alpha t)$ and Z_t satisfies the uncertain differential equation,

$$\mathrm{d}Z_t = \exp(-\alpha t)(\exp(\alpha t)Z_t)^\beta \mathrm{d}C_t = \exp((\beta-1)\alpha t)Z_t^\beta \mathrm{d}C_t.$$

Since $\beta \neq 1$, we have

$$\mathrm{d}Z_t^{1-\beta} = (1-\beta)\exp((\beta-1)\alpha t)\mathrm{d}C_t.$$

It follows from the fundamental theorem of uncertain calculus that

$$Z_t^{1-\beta} = Z_0^{1-\beta} + (1-\beta)\int_0^t \exp((\beta-1)\alpha s)\mathrm{d}C_s.$$

Since the initial value Z_0 is just X_0, we have

$$Z_t = \left(X_0^{1-\beta} + (1-\beta)\int_0^t \exp((\beta-1)\alpha s)\mathrm{d}C_s\right)^{1/(1-\beta)}.$$

Theorem 15.5 says the uncertain differential equation (15.30) has a solution $X_t = Y_t^{-1}Z_t$, i.e.,

$$X_t = \exp(\alpha t)\left(X_0^{1-\beta} + (1-\beta)\int_0^t \exp((\beta-1)\alpha s)\mathrm{d}C_s\right)^{1/(1-\beta)}.$$

Second Analytic Method

This subsection will introduce an analytic method for solving nonlinear uncertain differential equations like

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \sigma_t\mathrm{d}C_t \tag{15.31}$$

and

$$\mathrm{d}X_t = \alpha_t\mathrm{d}t + g(t, X_t)\mathrm{d}C_t. \tag{15.32}$$

Theorem 15.6 *(Yao [247]) Let f be a function of two variables and let σ_t be an integrable uncertain process. Then the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + \sigma_t\mathrm{d}C_t \tag{15.33}$$

has a solution

$$X_t = Y_t + Z_t \tag{15.34}$$

where

$$Y_t = \int_0^t \sigma_s\mathrm{d}C_s \tag{15.35}$$

and Z_t is the solution of the uncertain differential equation

$$\mathrm{d}Z_t = f(t, Y_t + Z_t)\mathrm{d}t \tag{15.36}$$

with initial value $Z_0 = X_0$.

Proof: At first, Y_t has an uncertain differential $\mathrm{d}Y_t = \sigma_t \mathrm{d}C_t$. It follows that

$$\mathrm{d}(X_t - Y_t) = \mathrm{d}X_t - \mathrm{d}Y_t = f(t, X_t)\mathrm{d}t + \sigma_t \mathrm{d}C_t - \sigma_t \mathrm{d}C_t.$$

That is,

$$\mathrm{d}(X_t - Y_t) = f(t, X_t)\mathrm{d}t.$$

Defining $Z_t = X_t - Y_t$, we obtain $X_t = Y_t + Z_t$ and $\mathrm{d}Z_t = f(t, Y_t + Z_t)\mathrm{d}t$. Furthermore, since $Y_0 = 0$, the initial value Z_0 is just X_0. The theorem is proved.

Example 15.7: Let α and σ be real numbers with $\alpha \neq 0$. Consider the uncertain differential equation

$$\mathrm{d}X_t = \alpha \exp(X_t)\mathrm{d}t + \sigma \mathrm{d}C_t. \tag{15.37}$$

At first, we have $Y_t = \sigma C_t$ and Z_t satisfies the uncertain differential equation,

$$\mathrm{d}Z_t = \alpha \exp(\sigma C_t + Z_t)\mathrm{d}t.$$

Since $\alpha \neq 0$, we have

$$\mathrm{d}\exp(-Z_t) = -\alpha \exp(\sigma C_t)\mathrm{d}t.$$

It follows from the fundamental theorem of uncertain calculus that

$$\exp(-Z_t) = \exp(-Z_0) - \alpha \int_0^t \exp(\sigma C_s)\mathrm{d}s.$$

Since the initial value Z_0 is just X_0, we have

$$Z_t = X_0 - \ln\left(1 - \alpha \int_0^t \exp(X_0 + \sigma C_s)\mathrm{d}s\right).$$

Hence

$$X_t = X_0 + \sigma C_t - \ln\left(1 - \alpha \int_0^t \exp(X_0 + \sigma C_s)\mathrm{d}s\right).$$

Theorem 15.7 *(Yao [247]) Let g be a function of two variables and let α_t be an integrable uncertain process. Then the uncertain differential equation*

$$\mathrm{d}X_t = \alpha_t \mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{15.38}$$

has a solution

$$X_t = Y_t + Z_t \tag{15.39}$$

where

$$Y_t = \int_0^t \alpha_s \mathrm{d}s \tag{15.40}$$

and Z_t is the solution of the uncertain differential equation

$$\mathrm{d}Z_t = g(t, Y_t + Z_t)\mathrm{d}C_t \tag{15.41}$$

with initial value $Z_0 = X_0$.

Proof: The uncertain process Y_t has an uncertain differential $\mathrm{d}Y_t = \alpha_t\mathrm{d}t$. It follows that

$$\mathrm{d}(X_t - Y_t) = \mathrm{d}X_t - \mathrm{d}Y_t = \alpha_t\mathrm{d}t + g(t, X_t)\mathrm{d}C_t - \alpha_t\mathrm{d}t.$$

That is,

$$\mathrm{d}(X_t - Y_t) = g(t, X_t)\mathrm{d}C_t.$$

Defining $Z_t = X_t - Y_t$, we obtain $X_t = Y_t + Z_t$ and $\mathrm{d}Z_t = g(t, Y_t + Z_t)\mathrm{d}C_t$. Furthermore, since $Y_0 = 0$, the initial value Z_0 is just X_0. The theorem is proved.

Example 15.8: Let α and σ be real numbers with $\sigma \neq 0$. Consider the uncertain differential equation

$$\mathrm{d}X_t = \alpha\mathrm{d}t + \sigma\exp(X_t)\mathrm{d}C_t. \tag{15.42}$$

At first, we have $Y_t = \alpha t$ and Z_t satisfies the uncertain differential equation,

$$\mathrm{d}Z_t = \sigma\exp(\alpha t + Z_t)\mathrm{d}C_t.$$

Since $\sigma \neq 0$, we have

$$\mathrm{d}\exp(-Z_t) = \sigma\exp(\alpha t)\mathrm{d}C_t.$$

It follows from the fundamental theorem of uncertain calculus that

$$\exp(-Z_t) = \exp(-Z_0) + \sigma\int_0^t \exp(\alpha s)\mathrm{d}C_s.$$

Since the initial value Z_0 is just X_0, we have

$$Z_t = X_0 - \ln\left(1 - \sigma\int_0^t \exp(X_0 + \alpha s)\mathrm{d}C_s\right).$$

Hence

$$X_t = X_0 + \alpha t - \ln\left(1 - \sigma\int_0^t \exp(X_0 + \alpha s)\mathrm{d}C_s\right).$$

15.3 Existence and Uniqueness

Theorem 15.8 *(Chen and Liu [12], Existence and Uniqueness Theorem) The uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{15.43}$$

has a unique solution if the coefficients $f(t, x)$ and $g(t, x)$ satisfy linear growth condition

$$|f(t, x)| + |g(t, x)| \le L(1 + |x|), \quad \forall x \in \Re, t \ge 0 \tag{15.44}$$

and Lipschitz condition

$$|f(t,x)-f(t,y)|+|g(t,x)-g(t,y)|\le L|x-y|,\quad \forall x,y\in\Re, t\ge 0 \quad (15.45)$$

for some constant L. Moreover, the solution is sample-continuous.

Proof: We first prove the existence of solution by a successive approximation method. Define $X_t^{(0)}=X_0$, and

$$X_t^{(n)}=X_0+\int_0^t f\left(s,X_s^{(n-1)}\right)\mathrm{d}s+\int_0^t g\left(s,X_s^{(n-1)}\right)\mathrm{d}C_s$$

for $n=1,2,\cdots$ and write

$$D_t^{(n)}(\gamma)=\max_{0\le s\le t}\left|X_s^{(n+1)}(\gamma)-X_s^{(n)}(\gamma)\right|$$

for each $\gamma\in\Gamma$. It follows from linear growth condition and Lipschitz condition that

$$\begin{aligned}D_t^{(0)}(\gamma)&=\max_{0\le s\le t}\left|\int_0^s f(v,X_0)\mathrm{d}v+\int_0^s g(v,X_0)\mathrm{d}C_v(\gamma)\right|\\&\le\int_0^t|f(v,X_0)|\,\mathrm{d}v+K_\gamma\int_0^t|g(v,X_0)|\,\mathrm{d}v\\&\le(1+|X_0|)L(1+K_\gamma)t\end{aligned}$$

where K_γ is the Lipschitz constant to the sample path $C_t(\gamma)$. In fact, by using the induction method, we may verify

$$D_t^{(n)}(\gamma)\le(1+|X_0|)\frac{L^{n+1}(1+K_\gamma)^{n+1}}{(n+1)!}t^{n+1}$$

for each n. This means that, for each $\gamma\in\Gamma$, the sample paths $X_t^{(k)}(\gamma)$ converges uniformly on any given time interval. Write the limit by $X_t(\gamma)$ that is just a solution of the uncertain differential equation because

$$X_t=X_0+\int_0^t f(s,X_s)\mathrm{d}s+\int_0^t g(s,X_s)\mathrm{d}s.$$

Next we prove that the solution is unique. Assume that both X_t and X_t^* are solutions of the uncertain differential equation. Then for each $\gamma\in\Gamma$, it follows from linear growth condition and Lipschitz condition that

$$|X_t(\gamma)-X_t^*(\gamma)|\le L(1+K_\gamma)\int_0^t|X_v(\gamma)-X_v^*(\gamma)|\mathrm{d}v.$$

By using Gronwall inequality, we obtain

$$|X_t(\gamma)-X_t^*(\gamma)|\le 0\cdot\exp(L(1+K_\gamma)t)=0.$$

Hence $X_t = X_t^*$. The uniqueness is verified. Finally, for each $\gamma \in \Gamma$, we have

$$|X_t(\gamma) - X_r(\gamma)| = \left| \int_r^t f(s, X_s(\gamma))\mathrm{d}s + \int_r^t g(s, X_s(\gamma))\mathrm{d}C_s(\gamma) \right| \to 0$$

as $r \to t$. Thus X_t is sample-continuous and the theorem is proved.

15.4 Stability

Definition 15.2 *(Liu [125]) An uncertain differential equation is said to be stable if for any two solutions X_t and Y_t, we have*

$$\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{|X_t - Y_t| < \varepsilon \text{ for all } t \geq 0\} = 1 \tag{15.46}$$

for any given number $\varepsilon > 0$.

Example 15.9: In order to illustrate the concept of stability, let us consider the uncertain differential equation

$$\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t. \tag{15.47}$$

It is clear that two solutions with initial values X_0 and Y_0 are

$$X_t = X_0 + at + bC_t,$$

$$Y_t = Y_0 + at + bC_t.$$

Then for any given number $\varepsilon > 0$, we have

$$\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{|X_t - Y_t| < \varepsilon \text{ for all } t \geq 0\} = \lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{|X_0 - Y_0| < \varepsilon\} = 1.$$

Hence the uncertain differential equation (15.47) is stable.

Example 15.10: Some uncertain differential equations are not stable. For example, consider

$$\mathrm{d}X_t = X_t\mathrm{d}t + b\mathrm{d}C_t. \tag{15.48}$$

It is clear that two solutions with different initial values X_0 and Y_0 are

$$X_t = \exp(t)X_0 + b\exp(t)\int_0^t \exp(-s)\mathrm{d}C_s,$$

$$Y_t = \exp(t)Y_0 + b\exp(t)\int_0^t \exp(-s)\mathrm{d}C_s.$$

Then for any given number $\varepsilon > 0$, we have

$$\begin{aligned} &\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{|X_t - Y_t| < \varepsilon \text{ for all } t \geq 0\} \\ = &\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{\exp(t)|X_0 - Y_0| < \varepsilon \text{ for all } t \geq 0\} = 0. \end{aligned}$$

Hence the uncertain differential equation (15.48) is unstable.

Theorem 15.9 *(Yao, Gao and Gao [243], Stability Theorem) The uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{15.49}$$

is stable if the coefficients $f(t,x)$ and $g(t,x)$ satisfy linear growth condition

$$|f(t,x)| + |g(t,x)| \le K(1+|x|), \quad \forall x \in \Re, t \ge 0 \tag{15.50}$$

for some constant K and strong Lipschitz condition

$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \le L(t)|x-y|, \quad \forall x, y \in \Re, t \ge 0 \tag{15.51}$$

for some bounded and integrable function $L(t)$ on $[0,+\infty)$.

Proof: Since $L(t)$ is bounded on $[0,+\infty)$, there is a constant R such that $L(t) \le R$ for any t. Then the strong Lipschitz condition (15.51) implies the following Lipschitz condition,

$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \le R|x-y|, \quad \forall x, y \in \Re, t \ge 0. \tag{15.52}$$

It follows from linear growth condition (15.50), Lipschitz condition (15.52) and the existence and uniqueness theorem that the uncertain differential equation (15.49) has a unique solution. Let X_t and Y_t be two solutions with initial values X_0 and Y_0, respectively. Then for each γ, we have

$$\begin{aligned}
\mathrm{d}|X_t(\gamma) - Y_t(\gamma)| &\le |f(t, X_t(\gamma)) - f(t, Y_t(\gamma))| + |g(t, X_t(\gamma)) - g(t, Y_t(\gamma))| \\
&\le L(t)|X_t(\gamma) - Y_t(\gamma)|\mathrm{d}t + L(t)K(\gamma)|X_t(\gamma) - Y_t(\gamma)|\mathrm{d}t \\
&= L(t)(1+K(\gamma))|X_t(\gamma) - Y_t(\gamma)|\mathrm{d}t
\end{aligned}$$

where $K(\gamma)$ is the Lipschitz constant of the sample path $C_t(\gamma)$. It follows that

$$|X_t(\gamma) - Y_t(\gamma)| \le |X_0 - Y_0| \exp\left((1+K(\gamma))\int_0^{+\infty} L(s)\mathrm{d}s\right).$$

Thus for any given $\varepsilon > 0$, we always have

$$\begin{aligned}
&\mathcal{M}\{|X_t - Y_t| < \varepsilon \text{ for all } t \ge 0\} \\
\ge\ &\mathcal{M}\left\{|X_0 - Y_0| \exp\left((1+K(\gamma))\int_0^{+\infty} L(s)\mathrm{d}s\right) < \varepsilon\right\}.
\end{aligned}$$

Since

$$\mathcal{M}\left\{|X_0 - Y_0| \exp\left((1+K(\gamma))\int_0^{+\infty} L(s)\mathrm{d}s\right) < \varepsilon\right\} \to 1$$

as $|X_0 - Y_0| \to 0$, we obtain

$$\lim_{|X_0 - Y_0| \to 0} \mathcal{M}\{|X_t - Y_t| < \varepsilon \text{ for all } t \ge 0\} = 1.$$

Hence the uncertain differential equation is stable.

Exercise 15.1: Suppose $u_{1t}, u_{2t}, v_{1t}, v_{2t}$ are bounded functions with respect to t such that

$$\int_0^{+\infty} |u_{1t}|\mathrm{d}t < +\infty, \quad \int_0^{+\infty} |v_{1t}|\mathrm{d}t < +\infty. \tag{15.53}$$

Show that the linear uncertain differential equation

$$\mathrm{d}X_t = (u_{1t}X_t + u_{2t})\mathrm{d}t + (v_{1t}X_t + v_{2t})\mathrm{d}C_t \tag{15.54}$$

is stable.

15.5 α-Path

Definition 15.3 *(Yao and Chen [246]) Let α be a number with $0 < \alpha < 1$. An uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{15.55}$$

is said to have an α-path X_t^α if it solves the corresponding ordinary differential equation

$$\mathrm{d}X_t^\alpha = f(t, X_t^\alpha)\mathrm{d}t + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t \tag{15.56}$$

where $\Phi^{-1}(\alpha)$ is the inverse standard normal uncertainty distribution, i.e.,

$$\Phi^{-1}(\alpha) = \frac{\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}. \tag{15.57}$$

Remark 15.2: Note that each α-path X_t^α is a real-valued function of time t, but is not necessarily one of sample paths. Furthermore, almost all α-paths are continuous functions with respect to time t.

Example 15.11: The uncertain differential equation $\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t$ with $X_0 = 0$ has an α-path

$$X_t^\alpha = at + |b|\Phi^{-1}(\alpha)t \tag{15.58}$$

where Φ^{-1} is the inverse standard normal uncertainty distribution.

Example 15.12: The uncertain differential equation $\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}C_t$ with $X_0 = 1$ has an α-path

$$X_t^\alpha = \exp\left(at + |b|\Phi^{-1}(\alpha)t\right) \tag{15.59}$$

where Φ^{-1} is the inverse standard normal uncertainty distribution.

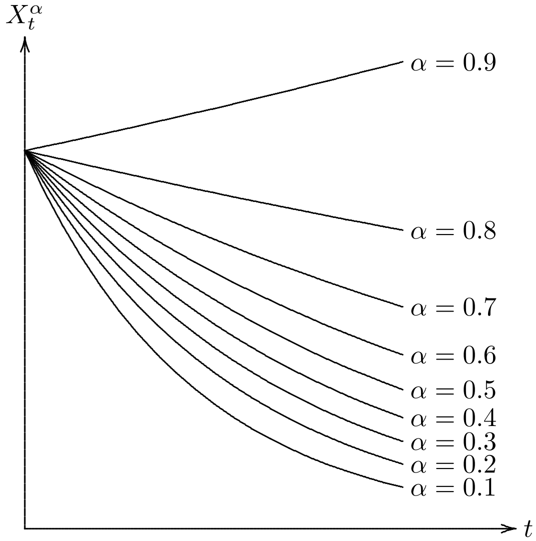


Figure 15.1: A Spectrum of α-Paths of $\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}C_t$. Reprinted from Liu [129].

15.6 Yao-Chen Formula

Yao-Chen formula relates uncertain differential equations and ordinary differential equations, just like that Feynman-Kac formula relates stochastic differential equations and partial differential equations.

Theorem 15.10 *(Yao-Chen Formula [246]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{15.60}$$

respectively. Then

$$\mathcal{M}\{X_t \le X_t^\alpha, \forall t\} = \alpha, \tag{15.61}$$

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1 - \alpha. \tag{15.62}$$

Proof: At first, for each α-path X_t^α, we divide the time interval into two parts,

$$T^+ = \left\{t \mid g\left(t, X_t^\alpha\right) \ge 0\right\},$$

$$T^- = \left\{t \mid g\left(t, X_t^\alpha\right) < 0\right\}.$$

It is obvious that $T^+ \cap T^- = \emptyset$ and $T^+ \cup T^- = [0, +\infty)$. Write

$$\Lambda_1^+ = \left\{\gamma \;\Big|\; \frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} \le \Phi^{-1}(\alpha) \text{ for any } t \in T^+\right\},$$

$$\Lambda_1^- = \left\{\gamma \mid \frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} \ge \Phi^{-1}(1-\alpha) \text{ for any } t \in T^-\right\}$$

where Φ^{-1} is the inverse standard normal uncertainty distribution. Since T^+ and T^- are disjoint sets and C_t has independent increments, we get

$$\mathcal{M}\{\Lambda_1^+\} = \alpha, \quad \mathcal{M}\{\Lambda_1^-\} = \alpha, \quad \mathcal{M}\{\Lambda_1^+ \cap \Lambda_1^-\} = \alpha.$$

For any $\gamma \in \Lambda_1^+ \cap \Lambda_1^-$, we always have

$$g(t, X_t(\gamma))\frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} \le |g(t, X_t^\alpha)|\Phi^{-1}(\alpha), \ \forall t.$$

Hence $X_t(\gamma) \le X_t^\alpha$ for all t and

$$\mathcal{M}\{X_t \le X_t^\alpha, \forall t\} \ge \mathcal{M}\{\Lambda_1^+ \cap \Lambda_1^-\} = \alpha. \tag{15.63}$$

On the other hand, let us define

$$\Lambda_2^+ = \left\{\gamma \mid \frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} > \Phi^{-1}(\alpha) \text{ for any } t \in T^+\right\},$$

$$\Lambda_2^- = \left\{\gamma \mid \frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} < \Phi^{-1}(1-\alpha) \text{ for any } t \in T^-\right\}.$$

Since T^+ and T^- are disjoint sets and C_t has independent increments, we obtain

$$\mathcal{M}\{\Lambda_2^+\} = 1-\alpha, \quad \mathcal{M}\{\Lambda_2^-\} = 1-\alpha, \quad \mathcal{M}\{\Lambda_2^+ \cap \Lambda_2^-\} = 1-\alpha.$$

For any $\gamma \in \Lambda_2^+ \cap \Lambda_2^-$, we always have

$$g(t, X_t(\gamma))\frac{\mathrm{d}C_t(\gamma)}{\mathrm{d}t} > |g(t, X_t^\alpha)|\Phi^{-1}(\alpha), \ \forall t.$$

Hence $X_t(\gamma) > X_t^\alpha$ for all t and

$$\mathcal{M}\{X_t > X_t^\alpha, \forall t\} \ge \mathcal{M}\{\Lambda_2^+ \cap \Lambda_2^-\} = 1-\alpha. \tag{15.64}$$

Note that $\{X_t \le X_t^\alpha, \forall t\}$ and $\{X_t \not\le X_t^\alpha, \forall t\}$ are opposite events with each other. By using the duality axiom, we obtain

$$\mathcal{M}\{X_t \le X_t^\alpha, \forall t\} + \mathcal{M}\{X_t \not\le X_t^\alpha, \forall t\} = 1.$$

It follows from $\mathcal{M}\{X_t > X_t^\alpha, \forall t\} \subset \mathcal{M}\{X_t \not\le X_t^\alpha, \forall t\}$ and monotonicity theorem that

$$\mathcal{M}\{X_t \le X_t^\alpha, \forall t\} + \mathcal{M}\{X_t > X_t^\alpha, \forall t\} \le 1. \tag{15.65}$$

Thus (15.61) and (15.62) follow from (15.63), (15.64) and (15.65) immediately.

Remark 15.3: It is also shown that Yao-Chen formula may be written as

$$\mathcal{M}\{X_t < X_t^\alpha, \forall t\} = \alpha, \tag{15.66}$$

$$\mathcal{M}\{X_t \ge X_t^\alpha, \forall t\} = 1 - \alpha. \tag{15.67}$$

Please mention that $\{X_t < X_t^\alpha, \forall t\}$ and $\{X_t \ge X_t^\alpha, \forall t\}$ are disjoint events but not opposite. Generally speaking, their union is not the universal set, and it is possible that

$$\mathcal{M}\{(X_t < X_t^\alpha, \forall t) \cup (X_t \ge X_t^\alpha, \forall t)\} < 1. \tag{15.68}$$

However, for any α, it is always true that

$$\mathcal{M}\{X_t < X_t^\alpha, \forall t\} + \mathcal{M}\{X_t \ge X_t^\alpha, \forall t\} \equiv 1. \tag{15.69}$$

Uncertainty Distribution of Solution

Theorem 15.11 *(Yao and Chen [246]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{15.70}$$

respectively. Then the solution X_t has an inverse uncertainty distribution

$$\Psi_t^{-1}(\alpha) = X_t^\alpha. \tag{15.71}$$

Proof: Note that $\{X_t \le X_t^\alpha\} \supset \{X_s \le X_s^\alpha, \forall s\}$ holds. By using the monotonicity theorem and Yao-Chen formula, we obtain

$$\mathcal{M}\{X_t \le X_t^\alpha\} \ge \mathcal{M}\{X_s \le X_s^\alpha, \forall s\} = \alpha. \tag{15.72}$$

Similarly, we also have

$$\mathcal{M}\{X_t > X_t^\alpha\} \ge \mathcal{M}\{X_s > X_s^\alpha, \forall s\} = 1 - \alpha. \tag{15.73}$$

In addition, since $\{X_t \le X_t^\alpha\}$ and $\{X_t > X_t^\alpha\}$ are opposite events, the duality axiom makes

$$\mathcal{M}\{X_t \le X_t^\alpha\} + \mathcal{M}\{X_t > X_t^\alpha\} = 1. \tag{15.74}$$

It follows from (15.72), (15.73) and (15.74) that $\mathcal{M}\{X_t \le X_t^\alpha\} = \alpha$. The theorem is thus verified.

Exercise 15.2: Show that the solution of the uncertain differential equation $\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}C_t$ with $X_0 = 0$ has an inverse uncertainty distribution

$$\Psi_t^{-1}(\alpha) = at + |b|\Phi^{-1}(\alpha)t \tag{15.75}$$

where Φ^{-1} is the inverse standard normal uncertainty distribution.

Exercise 15.3: Show that the solution of the uncertain differential equation $\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}C_t$ with $X_0 = 1$ has an inverse uncertainty distribution

$$\Psi_t^{-1}(\alpha) = \exp\left(at + |b|\Phi^{-1}(\alpha)t\right) \tag{15.76}$$

where Φ^{-1} is the inverse standard normal uncertainty distribution.

Expected Value of Solution

Theorem 15.12 *(Yao and Chen [246]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{15.77}$$

respectively. Then for any monotone (increasing or decreasing) function J, we have

$$E[J(X_t)] = \int_0^1 J(X_t^\alpha)\mathrm{d}\alpha. \tag{15.78}$$

Proof: At first, it follows from Yao-Chen formula that X_t has an uncertainty distribution $\Psi_t^{-1}(\alpha) = X_t^\alpha$. Next, we may have a monotone function become a strictly monotone function by a small perturbation. When J is a strictly increasing function, it follows from Theorem 2.9 that $J(X_t)$ has an inverse uncertainty distribution

$$\Upsilon_t^{-1}(\alpha) = J(X_t^\alpha).$$

Thus we have

$$E[J(X_t)] = \int_0^1 \Upsilon_t^{-1}(\alpha)\mathrm{d}\alpha = \int_0^1 J(X_t^\alpha)\mathrm{d}\alpha.$$

When J is a strictly decreasing function, it follows from Theorem 2.16 that $J(X_t)$ has an inverse uncertainty distribution

$$\Upsilon_t^{-1}(\alpha) = J(X_t^{1-\alpha}).$$

Thus we have

$$E[J(X_t)] = \int_0^1 \Upsilon_t^{-1}(\alpha)\mathrm{d}\alpha = \int_0^1 J(X_t^{1-\alpha})\mathrm{d}\alpha = \int_0^1 J(X_t^\alpha)\mathrm{d}\alpha.$$

The theorem is thus proved.

Exercise 15.4: Let X_t and X_t^α be the solution and α-path of some uncertain differential equation. Show that

$$E[X_t] = \int_0^1 X_t^\alpha \mathrm{d}\alpha, \tag{15.79}$$

$$E[(X_t - K)^+] = \int_0^1 (X_t^\alpha - K)^+\mathrm{d}\alpha, \tag{15.80}$$

$$E[(K - X_t)^+] = \int_0^1 (K - X_t^\alpha)^+\mathrm{d}\alpha. \tag{15.81}$$

Extreme Value of Solution

Theorem 15.13 *(Yao [244]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{15.82}$$

respectively. Then for any time $s > 0$ and strictly increasing function $J(x)$, the supremum

$$\sup_{0\le t\le s} J(X_t) \tag{15.83}$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \sup_{0\le t\le s} J(X_t^\alpha); \tag{15.84}$$

and the infimum

$$\inf_{0\le t\le s} J(X_t) \tag{15.85}$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \inf_{0\le t\le s} J(X_t^\alpha). \tag{15.86}$$

Proof: Since $J(x)$ is a strictly increasing function with respect to x, it is always true that

$$\left\{\sup_{0\le t\le s} J(X_t) \le \sup_{0\le t\le s} J(X_t^\alpha)\right\} \supset \{X_t \le X_t^\alpha, \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\left\{\sup_{0\le t\le s} J(X_t) \le \sup_{0\le t\le s} J(X_t^\alpha)\right\} \ge \mathcal{M}\{X_t \le X_t^\alpha, \forall t\} = \alpha. \tag{15.87}$$

Similarly, we have

$$\mathcal{M}\left\{\sup_{0\le t\le s} J(X_t) > \sup_{0\le t\le s} J(X_t^\alpha)\right\} \ge \mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1-\alpha. \tag{15.88}$$

It follows from (15.87), (15.88) and the duality axiom that

$$\mathcal{M}\left\{\sup_{0\le t\le s} J(X_t) \le \sup_{0\le t\le s} J(X_t^\alpha)\right\} = \alpha \tag{15.89}$$

which proves (15.84). Next, it is easy to verify that

$$\left\{\inf_{0\le t\le s} J(X_t) \le \inf_{0\le t\le s} J(X_t^\alpha)\right\} \supset \{X_t \le X_t^\alpha, \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\left\{\inf_{0\le t\le s} J(X_t) \le \inf_{0\le t\le s} J(X_t^\alpha)\right\} \ge \mathcal{M}\{X_t \le X_t^\alpha, \forall t\} = \alpha. \tag{15.90}$$

Similarly, we have

$$\mathcal{M}\left\{\inf_{0\le t\le s} J(X_t) > \inf_{0\le t\le s} J(X_t^\alpha)\right\} \ge \mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1-\alpha. \tag{15.91}$$

It follows from (15.90), (15.91) and the duality axiom that

$$\mathcal{M}\left\{\inf_{0\le t\le s} J(X_t) \le \inf_{0\le t\le s} J(X_t^\alpha)\right\} = \alpha \tag{15.92}$$

which proves (15.86). The theorem is thus verified.

Exercise 15.5: Let r and K be real numbers. Show that the supremum

$$\sup_{0\le t\le s} \exp(-rt)(X_t - K)$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \sup_{0\le t\le s} \exp(-rt)(X_t^\alpha - K)$$

for any given time $s > 0$.

Theorem 15.14 *(Yao [244]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{15.93}$$

respectively. Then for any time $s > 0$ and strictly decreasing function $J(x)$, the supremum

$$\sup_{0\le t\le s} J(X_t) \tag{15.94}$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \sup_{0\le t\le s} J(X_t^{1-\alpha}); \tag{15.95}$$

and the infimum

$$\inf_{0\le t\le s} J(X_t) \tag{15.96}$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \inf_{0\le t\le s} J(X_t^{1-\alpha}). \tag{15.97}$$

Proof: Since $J(x)$ is a strictly decreasing function with respect to x, it is always true that

$$\left\{\sup_{0\le t\le s} J(X_t) \le \sup_{0\le t\le s} J(X_t^{1-\alpha})\right\} \supset \{X_t \ge X_t^{1-\alpha}, \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\left\{\sup_{0\le t\le s} J(X_t) \le \sup_{0\le t\le s} J(X_t^{1-\alpha})\right\} \ge \mathcal{M}\{X_t \ge X_t^{1-\alpha}, \forall t\} = \alpha. \quad (15.98)$$

Similarly, we have

$$\mathcal{M}\left\{\sup_{0\le t\le s} J(X_t) > \sup_{0\le t\le s} J(X_t^{1-\alpha})\right\} \ge \mathcal{M}\{X_t < X_t^{1-\alpha}, \forall t\} = 1-\alpha. \quad (15.99)$$

It follows from (15.98), (15.99) and the duality axiom that

$$\mathcal{M}\left\{\sup_{0\le t\le s} J(X_t) \le \sup_{0\le t\le s} J(X_t^{1-\alpha})\right\} = \alpha \quad (15.100)$$

which proves (15.95). Next, it is easy to verify that

$$\left\{\inf_{0\le t\le s} J(X_t) \le \inf_{0\le t\le s} J(X_t^{1-\alpha})\right\} \supset \{X_t \ge X_t^{1-\alpha}, \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\left\{\inf_{0\le t\le s} J(X_t) \le \inf_{0\le t\le s} J(X_t^{1-\alpha})\right\} \ge \mathcal{M}\{X_t \ge X_t^{1-\alpha}, \forall t\} = \alpha. \quad (15.101)$$

Similarly, we have

$$\mathcal{M}\left\{\inf_{0\le t\le s} J(X_t) > \inf_{0\le t\le s} J(X_t^{1-\alpha})\right\} \ge \mathcal{M}\{X_t < X_t^{1-\alpha}, \forall t\} = 1-\alpha. \quad (15.102)$$

It follows from (15.101), (15.102) and the duality axiom that

$$\mathcal{M}\left\{\inf_{0\le t\le s} J(X_t) \le \inf_{0\le t\le s} J(X_t^{1-\alpha})\right\} = \alpha \quad (15.103)$$

which proves (15.97). The theorem is thus verified.

Exercise 15.6: Let r and K be real numbers. Show that the supremum

$$\sup_{0\le t\le s} \exp(-rt)(K - X_t)$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \sup_{0\le t\le s} \exp(-rt)(K - X_t^{1-\alpha})$$

for any given time $s > 0$.

First Hitting Time of Solution

Theorem 15.15 *(Yao [244]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{15.104}$$

with an initial value X_0, respectively. Then for any given level z and strictly increasing function $J(x)$, the first hitting time τ_z that $J(X_t)$ reaches z has an uncertainty distribution

$$\Psi(s) = \begin{cases} 1 - \inf\left\{\alpha \;\Big|\; \sup\limits_{0\le t\le s} J(X_t^\alpha) \ge z\right\}, & \text{if } z > J(X_0) \\ \sup\left\{\alpha \;\Big|\; \inf\limits_{0\le t\le s} J(X_t^\alpha) \le z\right\}, & \text{if } z < J(X_0). \end{cases} \tag{15.105}$$

Proof: At first, assume $z > J(X_0)$ and write

$$\alpha_0 = \inf\left\{\alpha \;\Big|\; \sup_{0\le t\le s} J(X_t^\alpha) \ge z\right\}.$$

Then we have

$$\sup_{0\le t\le s} J(X_t^{\alpha_0}) = z,$$

$$\{\tau_z \le s\} = \left\{\sup_{0\le t\le s} J(X_t) \ge z\right\} \supset \{X_t \ge X_t^{\alpha_0}, \forall t\},$$

$$\{\tau_z > s\} = \left\{\sup_{0\le t\le s} J(X_t) < z\right\} \supset \{X_t < X_t^{\alpha_0}, \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\{\tau_z \le s\} \ge \mathcal{M}\{X_t \ge X_t^{\alpha_0}, \forall t\} = 1 - \alpha_0,$$

$$\mathcal{M}\{\tau_z > s\} \ge \mathcal{M}\{X_t < X_t^{\alpha_0}, \forall t\} = \alpha_0.$$

It follows from $\mathcal{M}\{\tau_z \le s\} + \mathcal{M}\{\tau_z > s\} = 1$ that $\mathcal{M}\{\tau_z \le s\} = 1 - \alpha_0$. Hence the first hitting time τ_z has an uncertainty distribution

$$\Psi(s) = \mathcal{M}\{\tau_z \le s\} = 1 - \inf\left\{\alpha \;\Big|\; \sup_{0\le t\le s} J(X_t^\alpha) \ge z\right\}.$$

Similarly, assume $z < J(X_0)$ and write

$$\alpha_0 = \sup\left\{\alpha \;\Big|\; \inf_{0\le t\le s} J(X_t^\alpha) \le z\right\}.$$

Then we have

$$\inf_{0\le t\le s} J(X_t^{\alpha_0}) = z,$$

$$\{\tau_z \le s\} = \left\{\inf_{0\le t\le s} J(X_t) \le z\right\} \supset \{X_t \le X_t^{\alpha_0}, \forall t\},$$

$$\{\tau_z > s\} = \left\{\inf_{0\le t\le s} J(X_t) > z\right\} \supset \{X_t > X_t^{\alpha_0}, \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\{\tau_z \le s\} \ge \mathcal{M}\{X_t \le X_t^{\alpha_0}, \forall t\} = \alpha_0,$$

$$\mathcal{M}\{\tau_z > s\} \ge \mathcal{M}\{X_t > X_t^{\alpha_0}, \forall t\} = 1 - \alpha_0.$$

It follows from $\mathcal{M}\{\tau_z \le s\} + \mathcal{M}\{\tau_z > s\} = 1$ that $\mathcal{M}\{\tau_z \le s\} = \alpha_0$. Hence the first hitting time τ_z has an uncertainty distribution

$$\Psi(s) = \mathcal{M}\{\tau_z \le s\} = \sup\left\{\alpha \;\middle|\; \inf_{0\le t\le s} J(X_t^\alpha) \le z\right\}.$$

The theorem is verified.

Theorem 15.16 *(Yao [244]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t \tag{15.106}$$

with an initial value X_0, respectively. Then for any given level z and strictly decreasing function $J(x)$, the first hitting time τ_z that $J(X_t)$ reaches z has an uncertainty distribution

$$\Psi(s) = \begin{cases} \sup\left\{\alpha \;\middle|\; \sup\limits_{0\le t\le s} J(X_t^\alpha) \ge z\right\}, & \text{if } z > J(X_0) \\ 1 - \inf\left\{\alpha \;\middle|\; \inf\limits_{0\le t\le s} J(X_t^\alpha) \le z\right\}, & \text{if } z < J(X_0). \end{cases} \tag{15.107}$$

Proof: At first, assume $z > J(X_0)$ and write

$$\alpha_0 = \sup\left\{\alpha \;\middle|\; \sup_{0\le t\le s} J(X_t^\alpha) \ge z\right\}.$$

Then we have

$$\sup_{0\le t\le s} J(X_t^{\alpha_0}) = z,$$

$$\{\tau_z \le s\} = \left\{\sup_{0\le t\le s} J(X_t) \ge z\right\} \supset \{X_t \le X_t^{\alpha_0}, \forall t\},$$

$$\{\tau_z > s\} = \left\{\sup_{0\le t\le s} J(X_t) < z\right\} \supset \{X_t > X_t^{\alpha_0}, \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\{\tau_z \le s\} \ge \mathcal{M}\{X_t \le X_t^{\alpha_0}, \forall t\} = \alpha_0,$$

$$\mathcal{M}\{\tau_z > s\} \ge \mathcal{M}\{X_t > X_t^{\alpha_0}, \forall t\} = 1 - \alpha_0.$$

It follows from $\mathcal{M}\{\tau_z \le s\} + \mathcal{M}\{\tau_z > s\} = 1$ that $\mathcal{M}\{\tau_z \le s\} = \alpha_0$. Hence the first hitting time τ_z has an uncertainty distribution

$$\Psi(s) = \mathcal{M}\{\tau_z \le s\} = \sup\left\{\alpha \;\middle|\; \sup_{0\le t\le s} J(X_t^\alpha) \ge z\right\}.$$

Similarly, assume $z < J(X_0)$ and write

$$\alpha_0 = \inf\left\{\alpha \;\middle|\; \inf_{0\le t\le s} J(X_t^\alpha) \le z\right\}.$$

Then we have

$$\inf_{0\le t\le s} J(X_t^{\alpha_0}) = z,$$

$$\{\tau_z \le s\} = \left\{\inf_{0\le t\le s} J(X_t) \le z\right\} \supset \{X_t \ge X_t^{\alpha_0}, \forall t\},$$

$$\{\tau_z > s\} = \left\{\inf_{0\le t\le s} J(X_t) > z\right\} \supset \{X_t < X_t^{\alpha_0}, \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\{\tau_z \le s\} \ge \mathcal{M}\{X_t \ge X_t^{\alpha_0}, \forall t\} = 1 - \alpha_0,$$

$$\mathcal{M}\{\tau_z > s\} \ge \mathcal{M}\{X_t < X_t^{\alpha_0}, \forall t\} = \alpha_0.$$

It follows from $\mathcal{M}\{\tau_z \le s\} + \mathcal{M}\{\tau_z > s\} = 1$ that $\mathcal{M}\{\tau_z \le s\} = 1 - \alpha_0$. Hence the first hitting time τ_z has an uncertainty distribution

$$\Psi(s) = \mathcal{M}\{\tau_z \le s\} = 1 - \inf\left\{\alpha \;\middle|\; \inf_{0\le t\le s} J(X_t^\alpha) \le z\right\}.$$

The theorem is verified.

Time Integral of Solution

Theorem 15.17 *(Yao [244]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{15.108}$$

respectively. Then for any time $s > 0$ and strictly increasing function $J(x)$, the time integral

$$\int_0^s J(X_t)\mathrm{d}t \tag{15.109}$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \int_0^s J(X_t^\alpha)\mathrm{d}t. \tag{15.110}$$

Proof: Since $J(x)$ is a strictly increasing function with respect to x, it is always true that

$$\left\{\int_0^s J(X_t)\mathrm{d}t \le \int_0^s J(X_t^\alpha)\mathrm{d}t\right\} \supset \{J(X_t) \le J(X_t^\alpha), \forall t\} \supset \{X_t \le X_t^\alpha, \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\left\{\int_0^s J(X_t)\mathrm{d}t \le \int_0^s J(X_t^\alpha)\mathrm{d}t\right\} \ge \mathcal{M}\{X_t \le X_t^\alpha, \forall t\} = \alpha. \tag{15.111}$$

Similarly, we have

$$\mathcal{M}\left\{\int_0^s J(X_t)\mathrm{d}t > \int_0^s J(X_t^\alpha)\mathrm{d}t\right\} \ge \mathcal{M}\{X_t > X_t^\alpha, \forall t\} = 1-\alpha. \tag{15.112}$$

It follows from (15.111), (15.112) and the duality axiom that

$$\mathcal{M}\left\{\int_0^s J(X_t)\mathrm{d}t \le \int_0^s J(X_t^\alpha)\mathrm{d}t\right\} = \alpha. \tag{15.113}$$

The theorem is thus verified.

Exercise 15.7: Let r and K be real numbers. Show that the time integral

$$\int_0^s \exp(-rt)(X_t - K)\mathrm{d}t$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \int_0^s \exp(-rt)(X_t^\alpha - K)\mathrm{d}t$$

for any given time $s > 0$.

Theorem 15.18 *(Yao [244]) Let X_t and X_t^α be the solution and α-path of the uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{15.114}$$

respectively. Then for any time $s > 0$ and strictly decreasing function $J(x)$, the time integral

$$\int_0^s J(X_t)\mathrm{d}t \tag{15.115}$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \int_0^s J(X_t^{1-\alpha})\mathrm{d}t. \tag{15.116}$$

Proof: Since $J(x)$ is a strictly decreasing function with respect to x, it is always true that

$$\left\{\int_0^s J(X_t)\mathrm{d}t \le \int_0^s J(X_t^{1-\alpha})\mathrm{d}t\right\} \supset \{X_t \ge X_t^{1-\alpha},\ \forall t\}.$$

By using Yao-Chen formula, we obtain

$$\mathcal{M}\left\{\int_0^s J(X_t)\mathrm{d}t \le \int_0^s J(X_t^{1-\alpha})\mathrm{d}t\right\} \ge \mathcal{M}\{X_t \ge X_t^{1-\alpha},\ \forall t\} = \alpha. \quad (15.117)$$

Similarly, we have

$$\mathcal{M}\left\{\int_0^s J(X_t)\mathrm{d}t > \int_0^s J(X_t^{1-\alpha})\mathrm{d}t\right\} \ge \mathcal{M}\{X_t < X_t^{1-\alpha},\ \forall t\} = 1-\alpha. \quad (15.118)$$

It follows from (15.117), (15.118) and the duality axiom that

$$\mathcal{M}\left\{\int_0^s J(X_t)\mathrm{d}t \le \int_0^s J(X_t^{1-\alpha})\mathrm{d}t\right\} = \alpha. \quad (15.119)$$

The theorem is thus verified.

Exercise 15.8: Let r and K be real numbers. Show that the time integral

$$\int_0^s \exp(-rt)(K - X_t)\mathrm{d}t$$

has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \int_0^s \exp(-rt)(K - X_t^{1-\alpha})\mathrm{d}t$$

for any given time $s > 0$.

15.7 Numerical Methods

It is almost impossible to find analytic solutions for general uncertain differential equations. This fact provides a motivation to design some numerical methods to solve the uncertain differential equation

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t. \quad (15.120)$$

In order to do so, a key point is to obtain a spectrum of α-paths of the uncertain differential equation. For this purpose, Yao and Chen [246] designed a Euler method:

Step 1. Fix α on $(0, 1)$.

Step 2. Solve $\mathrm{d}X_t^\alpha = f(t, X_t^\alpha)\mathrm{d}t + |g(t, X_t^\alpha)|\Phi^{-1}(\alpha)\mathrm{d}t$ by any method of ordinary differential equation and obtain the α-path X_t^α, for example, by using the recursion formula

$$X_{i+1}^\alpha = X_i^\alpha + f(t_i, X_i^\alpha)h + |g(t_i, X_i^\alpha)|\Phi^{-1}(\alpha)h \tag{15.121}$$

where Φ^{-1} is the inverse standard normal uncertainty distribution and h is the step length.

Step 3. The α-path X_t^α is obtained.

Remark 15.4: Shen and Yao [209] designed a Runge-Kutta method that replaces the recursion formula (15.121) with

$$X_{i+1}^\alpha = X_i^\alpha + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{15.122}$$

where

$$k_1 = f(t_i, X_i^\alpha) + |g(t_i, X_i^\alpha)|\Phi^{-1}(\alpha), \tag{15.123}$$

$$k_2 = f(t_i + h/2, X_i^\alpha + h^2k_1/2) + |g(t_i + h/2, X_i^\alpha + h^2k_1/2)|\Phi^{-1}(\alpha), \tag{15.124}$$

$$k_3 = f(t_i + h/2, X_i^\alpha + h^2k_2/2) + |g(t_i + h/2, X_i^\alpha + h^2k_2/2)|\Phi^{-1}(\alpha), \tag{15.125}$$

$$k_4 = f(t_i + h, X_i^\alpha + h^2k_3) + |g(t_i + h, X_i^\alpha + h^2k_3)|\Phi^{-1}(\alpha). \tag{15.126}$$

Example 15.13: In order to illustrate the numerical method, let us consider an uncertain differential equation

$$\mathrm{d}X_t = (t - X_t)\mathrm{d}t + \sqrt{(1 + X_t)}\mathrm{d}C_t, \quad X_0 = 1. \tag{15.127}$$

The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may solve this equation successfully and obtain all α-paths of the uncertain differential equation. Furthermore, we may get

$$E[X_1] \approx 0.870. \tag{15.128}$$

Example 15.14: Now we consider a nonlinear uncertain differential equation

$$\mathrm{d}X_t = \sqrt{X_t}\mathrm{d}t + (1 - t)X_t\mathrm{d}C_t, \quad X_0 = 1. \tag{15.129}$$

Note that $(1 - t)X_t$ takes not only positive values but also negative values. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) may obtain all α-paths of the uncertain differential equation. Furthermore, we may get

$$E[(X_2 - 3)^+] \approx 2.845. \tag{15.130}$$

15.8 Bibliographic Notes

The study of uncertain differential equation was pioneered by Liu [123] in 2008. This work was immediately followed upon by many researchers. Nowadays, the uncertain differential equation has achieved fruitful results in both theory and practice.

The existence and uniqueness theorem of solution of uncertain differential equation was first proved by Chen and Liu [12] under linear growth condition and Lipschitz continuous condition. The theorem was verified again by Gao [53] under local linear growth condition and local Lipschitz continuous condition.

The first concept of stability of uncertain differential equation was presented by Liu [125], and some stability theorems were proved by Yao, Gao and Gao [243]. Following that, different types of stability of uncertain differential equations were explored, for example, stability in mean (Yao and Sheng [252]), stability in moment (Sheng and Wang [210]), almost sure stability (Liu, Ke and Fei [143]), and exponential stability (Sheng [214]).

In order to solve uncertain differential equations, Chen and Liu [12] obtained an analytic solution to linear uncertain differential equations. In addition, Liu [148] and Yao [247] presented a spectrum of analytic methods to solve some special classes of nonlinear uncertain differential equations.

More importantly, Yao and Chen [246] showed that the solution of an uncertain differential equation can be represented by a family of solutions of ordinary differential equations, thus relating uncertain differential equations and ordinary differential equations. On the basis of Yao-Chen formula, Yao [244] presented some formulas to calculate extreme value, first hitting time, and time integral of solution of uncertain differential equation. Furthermore, some numerical methods for solving general uncertain differential equations were designed among others by Yao and Chen [246] and Shen and Yao [209].

Uncertain differential equation was extended by many researchers. For example, uncertain delay differential equation was studied among others by Barbacioru [4], Ge and Zhu [54], and Liu and Fei [142]. In addition, uncertain differential equation with jumps was suggested by Yao [241], and backward uncertain differential equation was discussed by Ge and Zhu [55].

Uncertain differential equation has been widely applied in many fields such as uncertain finance (Liu [134]), uncertain optimal control (Zhu [284]), and uncertain differential game (Yang and Gao [238]).

Chapter 16

Uncertain Finance

This chapter will introduce uncertain stock model, uncertain interest rate model, and uncertain currency model by using the tool of uncertain differential equation.

16.1 Uncertain Stock Model

Liu [125] supposed that the stock price follows an uncertain differential equation and presented an *uncertain stock model* in which the bond price X_t and the stock price Y_t are determined by

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_t = eY_t\mathrm{d}t + \sigma Y_t\mathrm{d}C_t \end{cases} \tag{16.1}$$

where r is the riskless interest rate, e is the log-drift, σ is the log-diffusion, and C_t is a canonical Liu process. Note that the bond price is $X_t = X_0\exp(rt)$ and the stock price is

$$Y_t = Y_0\exp(et + \sigma C_t) \tag{16.2}$$

whose inverse uncertainty distribution is

$$\Phi_t^{-1}(\alpha) = Y_0\exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right). \tag{16.3}$$

European Option

Definition 16.1 *A European call option is a contract that gives the holder the right to buy a stock at an expiration time s for a strike price K.*

The payoff from a European call option is $(Y_s - K)^+$ since the option is rationally exercised if and only if $Y_s > K$. Considering the time value of money resulted from the bond, the present value of the payoff is $\exp(-rs)(Y_s - K)^+$.

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5_17

Hence the European call option price should be the expected present value of the payoff.

Definition 16.2 *Assume a European call option has a strike price K and an expiration time s. Then the European call option price is*

$$f_c = \exp(-rs)E[(Y_s - K)^+]. \tag{16.4}$$

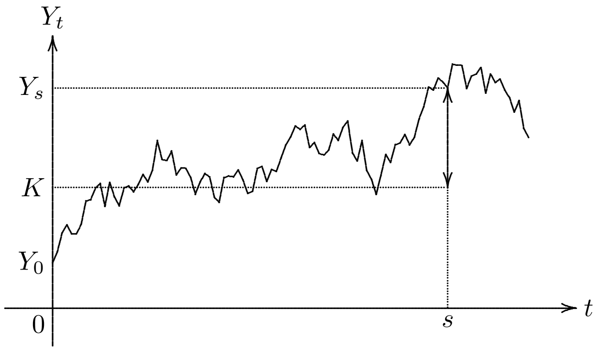


Figure 16.1: Payoff $(Y_s - K)^+$ from European Call Option

Theorem 16.1 *(Liu [125]) Assume a European call option for the uncertain stock model (16.1) has a strike price K and an expiration time s. Then the European call option price is*

$$f_c = \exp(-rs)\int_0^1 \left(Y_0 \exp\left(es + \frac{\sigma s\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right)^+ \mathrm{d}\alpha. \tag{16.5}$$

Proof: Since $(Y_s - K)^+$ is an increasing function with respect to Y_s, it has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \left(Y_0 \exp\left(es + \frac{\sigma s\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right)^+.$$

It follows from Definition 16.2 that the European call option price formula is just (16.5).

Remark 16.1: It is clear that the European call option price is a decreasing function of interest rate r. That is, the European call option will devaluate if the interest rate is raised; and the European call option will appreciate in value if the interest rate is reduced. In addition, the European call option price is also a decreasing function of the strike price K.

Example 16.1: Assume the interest rate $r = 0.08$, the log-drift $e = 0.06$, the log-diffusion $\sigma = 0.32$, the initial price $Y_0 = 20$, the strike price $K = 25$ and the expiration time $s = 2$. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields the European call option price

$$f_c = 6.91.$$

Definition 16.3 *A European put option is a contract that gives the holder the right to sell a stock at an expiration time s for a strike price K.*

The payoff from a European put option is $(K-Y_s)^+$ since the option is rationally exercised if and only if $Y_s < K$. Considering the time value of money resulted from the bond, the present value of this payoff is $\exp(-rs)(K-Y_s)^+$. Hence the European put option price should be the expected present value of the payoff.

Definition 16.4 *Assume a European put option has a strike price K and an expiration time s. Then the European put option price is*

$$f_p = \exp(-rs)E[(K - Y_s)^+]. \tag{16.6}$$

Theorem 16.2 *(Liu [125]) Assume a European put option for the uncertain stock model (16.1) has a strike price K and an expiration time s. Then the European put option price is*

$$f_p = \exp(-rs)\int_0^1 \left(K - Y_0 \exp\left(es + \frac{\sigma s\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha. \tag{16.7}$$

Proof: Since $(K - Y_s)^+$ is a decreasing function with respect to Y_s, it has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \left(Y_0 \exp\left(es + \frac{\sigma s\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right) - K\right)^+.$$

It follows from Definition 16.4 that the European put option price is

$$\begin{aligned} f_p &= \exp(-rs)\int_0^1 \left(K - Y_0 \exp\left(es + \frac{\sigma s\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right)^+ \mathrm{d}\alpha \\ &= \exp(-rs)\int_0^1 \left(K - Y_0 \exp\left(es + \frac{\sigma s\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha. \end{aligned}$$

The European put option price formula is verified.

Remark 16.2: It is easy to verify that the option price is a decreasing function of the interest rate r, and is an increasing function of the strike price K.

Example 16.2: Assume the interest rate $r = 0.08$, the log-drift $e = 0.06$, the log-diffusion $\sigma = 0.32$, the initial price $Y_0 = 20$, the strike price $K = 25$ and the expiration time $s = 2$. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields the European put option price

$$f_p = 4.40.$$

American Option

Definition 16.5 *An American call option is a contract that gives the holder the right to buy a stock at any time prior to an expiration time s for a strike price K.*

It is clear that the payoff from an American call option is the supremum of $(Y_t - K)^+$ over the time interval $[0, s]$. Considering the time value of money resulted from the bond, the present value of this payoff is

$$\sup_{0 \le t \le s} \exp(-rt)(Y_t - K)^+. \tag{16.8}$$

Hence the American call option price should be the expected present value of the payoff.

Definition 16.6 *Assume an American call option has a strike price K and an expiration time s. Then the American call option price is*

$$f_c = E\left[\sup_{0 \le t \le s} \exp(-rt)(Y_t - K)^+\right]. \tag{16.9}$$

Theorem 16.3 *(Chen [13]) Assume an American call option for the uncertain stock model (16.1) has a strike price K and an expiration time s. Then the American call option price is*

$$f_c = \int_0^1 \sup_{0 \le t \le s} \exp(-rt)\left(Y_0 \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right) - K\right)^+ \mathrm{d}\alpha.$$

Proof: It follows from Theorem 15.13 that $\sup_{0 \le t \le s} \exp(-rt)(Y_t - K)^+$ has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \sup_{0 \le t \le s} \exp(-rt)\left(Y_0 \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi} \ln \frac{\alpha}{1-\alpha}\right) - K\right)^+.$$

Hence the American call option price formula follows from Definition 16.6 immediately.

Remark 16.3: It is easy to verify that the option price is a decreasing function with respect to either the interest rate r or the strike price K.

Example 16.3: Assume the interest rate $r = 0.08$, the log-drift $e = 0.06$, the log-diffusion $\sigma = 0.32$, the initial price $Y_0 = 40$, the strike price $K = 38$ and the expiration time $s = 2$. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields the American call option price

$$f_c = 19.8.$$

Definition 16.7 *An American put option is a contract that gives the holder the right to sell a stock at any time prior to an expiration time s for a strike price K.*

It is clear that the payoff from an American put option is the supremum of $(K-Y_t)^+$ over the time interval $[0, s]$. Considering the time value of money resulted from the bond, the present value of this payoff is

$$\sup_{0\le t\le s} \exp(-rt)(K - Y_t)^+. \tag{16.10}$$

Hence the American put option price should be the expected present value of the payoff.

Definition 16.8 *Assume an American put option has a strike price K and an expiration time s. Then the American put option price is*

$$f_p = E\left[\sup_{0\le t\le s} \exp(-rt)(K - Y_t)^+\right]. \tag{16.11}$$

Theorem 16.4 *(Chen [13]) Assume an American put option for the uncertain stock model (16.1) has a strike price K and an expiration time s. Then the American put option price is*

$$f_p = \int_0^1 \sup_{0\le t\le s} \exp(-rt)\left(K - Y_0 \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha.$$

Proof: It follows from Theorem 15.14 that $\sup_{0\le t\le s} \exp(-rt)(K - Y_t)^+$ has an inverse uncertainty distribution

$$\Psi_s^{-1}(\alpha) = \sup_{0\le t\le s} \exp(-rt)\left(K - Y_0 \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{1-\alpha}{\alpha}\right)\right)^+.$$

Hence the American put option price formula follows from Definition 16.8 immediately.

Remark 16.4: It is easy to verify that the option price is a decreasing function of the interest rate r, and is an increasing function of the strike price K.

Example 16.4: Assume the interest rate $r = 0.08$, the log-drift $e = 0.06$, the log-diffusion $\sigma = 0.32$, the initial price $Y_0 = 40$, the strike price $K = 38$ and the expiration time $s = 2$. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields the American put option price

$$f_p = 3.90.$$

Asian Option

Definition 16.9 *An Asian call option is a contract whose payoff at the expiration time s is*

$$\left(\frac{1}{s}\int_0^s Y_t \mathrm{d}t - K\right)^+ \tag{16.12}$$

where K is a strike price.

Considering the time value of money resulted from the bond, the present value of the payoff from an Asian call option is

$$\exp(-rs)\left(\frac{1}{s}\int_0^s Y_t \mathrm{d}t - K\right)^+. \tag{16.13}$$

Hence the Asian call option price should be the expected present value of the payoff.

Definition 16.10 *Assume an Asian call option has a strike price K and an expiration time s. Then the Asian call option price is*

$$f_c = \exp(-rs)E\left[\left(\frac{1}{s}\int_0^s Y_t \mathrm{d}t - K\right)^+\right]. \tag{16.14}$$

Theorem 16.5 *(Sun and Chen [218]) Assume an Asian call option for the uncertain stock model (16.1) has a strike price K and an expiration time s. Then the Asian call option price is*

$$f_c = \exp(-rs)\int_0^1 \left(\frac{Y_0}{s}\int_0^s \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\mathrm{d}t - K\right)^+ \mathrm{d}\alpha.$$

Proof: It follows from Theorem 15.17 that the inverse uncertainty distribution of time integral

$$\int_0^s Y_t \mathrm{d}t$$

is

$$\Psi_s^{-1}(\alpha) = Y_0\int_0^s \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\mathrm{d}t.$$

Hence the Asian call option price formula follows from Definition 16.10 immediately.

Definition 16.11 *An Asian put option is a contract whose payoff at the expiration time s is*

$$\left(K - \frac{1}{s}\int_0^s Y_t \mathrm{d}t\right)^+ \tag{16.15}$$

where K is a strike price.

Considering the time value of money resulted from the bond, the present value of the payoff from an Asian put option is

$$\exp(-rs)\left(K - \frac{1}{s}\int_0^s Y_t \mathrm{d}t\right)^+. \tag{16.16}$$

Hence the Asian put option price should be the expected present value of the payoff.

Definition 16.12 *Assume an Asian put option has a strike price K and an expiration time s. Then the Asian put option price is*

$$f_p = \exp(-rs)E\left[\left(K - \frac{1}{s}\int_0^s Y_t \mathrm{d}t\right)^+\right]. \tag{16.17}$$

Theorem 16.6 *(Sun and Chen [218]) Assume an Asian put option for the uncertain stock model (16.1) has a strike price K and an expiration time s. Then the Asian put option price is*

$$f_c = \exp(-rs)\int_0^1 \left(K - \frac{Y_0}{s}\int_0^s \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\mathrm{d}t\right)^+ \mathrm{d}\alpha.$$

Proof: It follows from Theorem 15.17 that the inverse uncertainty distribution of time integral

$$\int_0^s Y_t \mathrm{d}t$$

is

$$\Psi_s^{-1}(\alpha) = Y_0\int_0^s \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\mathrm{d}t.$$

Hence the Asian put option price formula follows from Definition 16.12 immediately.

General Stock Model

Generally, we may assume the stock price follows a general uncertain differential equation and obtain a *general stock model* in which the bond price X_t and the stock price Y_t are determined by

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_t = F(t, Y_t)\mathrm{d}t + G(t, Y_t)\mathrm{d}C_t \end{cases} \tag{16.18}$$

where r is the riskless interest rate, F and G are two functions, and C_t is a canonical Liu process.

Note that the α-path Y_t^α of the stock price Y_t can be calculated by some numerical methods. Assume the strike price is K and the expiration time is s. It follows from Definition 16.2 and Theorem 15.12 that the European call option price is

$$f_c = \exp(-rs)\int_0^1 (Y_s^\alpha - K)^+ \mathrm{d}\alpha. \tag{16.19}$$

It follows from Definition 16.4 and Theorem 15.12 that the European put option price is

$$f_p = \exp(-rs)\int_0^1 (K - Y_s^\alpha)^+ \mathrm{d}\alpha. \tag{16.20}$$

It follows from Definition 16.6 and Theorem 15.13 that the American call option price is

$$f_c = \int_0^1 \left[\sup_{0\le t\le s} \exp(-rt)(Y_t^\alpha - K)^+\right] \mathrm{d}\alpha. \tag{16.21}$$

It follows from Definition 16.8 and Theorem 15.14 that the American put option price is

$$f_p = \int_0^1 \left[\sup_{0\le t\le s} \exp(-rt)(K - Y_t^\alpha)^+\right] \mathrm{d}\alpha. \tag{16.22}$$

It follows from Definition 16.9 and Theorem 15.17 that the Asian call option price is

$$f_c = \exp(-rs)\int_0^1 \left[\left(\frac{1}{s}\int_0^s Y_t^\alpha \mathrm{d}t - K\right)^+\right] \mathrm{d}\alpha. \tag{16.23}$$

It follows from Definition 16.11 and Theorem 15.18 that the Asian put option price is

$$f_p = \exp(-rs)\int_0^1 \left[\left(K - \frac{1}{s}\int_0^s Y_t^\alpha \mathrm{d}t\right)^+\right] \mathrm{d}\alpha. \tag{16.24}$$

Multifactor Stock Model

Now we assume that there are multiple stocks whose prices are determined by multiple Liu processes. In this case, we have a *multifactor stock model* in which the bond price X_t and the stock prices Y_{it} are determined by

$$\begin{cases} \mathrm{d}X_t = rX_t\mathrm{d}t \\ \mathrm{d}Y_{it} = e_iY_{it}\mathrm{d}t + \displaystyle\sum_{j=1}^n \sigma_{ij}Y_{it}\mathrm{d}C_{jt}, \ i = 1, 2, \cdots, m \end{cases} \tag{16.25}$$

where r is the riskless interest rate, e_i are the log-drifts, σ_{ij} are the log-diffusions, C_{jt} are independent Liu processes, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$.

Portfolio Selection

For the multifactor stock model (16.25), we have the choice of $m+1$ different investments. At each time t we may choose a portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ (i.e., the investment fractions meeting $\beta_t + \beta_{1t} + \cdots + \beta_{mt} = 1$). Then the wealth Z_t at time t should follow the uncertain differential equation

$$\mathrm{d}Z_t = r\beta_t Z_t \mathrm{d}t + \sum_{i=1}^{m} e_i \beta_{it} Z_t \mathrm{d}t + \sum_{i=1}^{m}\sum_{j=1}^{n} \sigma_{ij}\beta_{it} Z_t \mathrm{d}C_{jt}. \tag{16.26}$$

That is,

$$Z_t = Z_0 \exp(rt) \exp\left(\int_0^t \sum_{i=1}^{m} (e_i - r)\beta_{is}\mathrm{d}s + \sum_{j=1}^{n}\int_0^t \sum_{i=1}^{m} \sigma_{ij}\beta_{is}\mathrm{d}C_{js}\right).$$

Portfolio selection problem is to find an optimal portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ such that the wealth Z_s is maximized in the sense of expected value.

No-Arbitrage

The stock model (16.25) is said to be *no-arbitrage* if there is no portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ such that for some time $s > 0$, we have

$$\mathcal{M}\{\exp(-rs)Z_s \geq Z_0\} = 1 \tag{16.27}$$

and

$$\mathcal{M}\{\exp(-rs)Z_s > Z_0\} > 0 \tag{16.28}$$

where Z_t is determined by (16.26) and represents the wealth at time t.

Theorem 16.7 *(Yao's No-Arbitrage Theorem [248]) The multifactor stock model (16.25) is no-arbitrage if and only if the system of linear equations*

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} e_1 - r \\ e_2 - r \\ \vdots \\ e_m - r \end{pmatrix} \tag{16.29}$$

has a solution, i.e., $(e_1 - r, e_2 - r, \cdots, e_m - r)$ is a linear combination of column vectors $(\sigma_{11}, \sigma_{21}, \cdots, \sigma_{m1})$, $(\sigma_{12}, \sigma_{22}, \cdots, \sigma_{m2})$, $\cdots$, $(\sigma_{1n}, \sigma_{2n}, \cdots, \sigma_{mn})$.

Proof: When the portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$ is accepted, the wealth at each time t is

$$Z_t = Z_0 \exp(rt) \exp\left(\int_0^t \sum_{i=1}^{m} (e_i - r)\beta_{is}\mathrm{d}s + \sum_{j=1}^{n}\int_0^t \sum_{i=1}^{m} \sigma_{ij}\beta_{is}\mathrm{d}C_{js}\right).$$

Thus

$$\ln(\exp(-rt)Z_t) - \ln Z_0 = \int_0^t \sum_{i=1}^m (e_i - r)\beta_{is}\mathrm{d}s + \sum_{j=1}^n \int_0^t \sum_{i=1}^m \sigma_{ij}\beta_{is}\mathrm{d}C_{js}$$

is a normal uncertain variable with expected value

$$\int_0^t \sum_{i=1}^m (e_i - r)\beta_{is}\mathrm{d}s$$

and variance

$$\left(\sum_{j=1}^n \int_0^t \left|\sum_{i=1}^m \sigma_{ij}\beta_{is}\right| \mathrm{d}s\right)^2.$$

Assume the system (16.29) has a solution. The argument breaks down into two cases. Case I: for any given time t and portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$, suppose

$$\sum_{j=1}^n \int_0^t \left|\sum_{i=1}^m \sigma_{ij}\beta_{is}\right| \mathrm{d}s = 0.$$

Then

$$\sum_{i=1}^m \sigma_{ij}\beta_{is} = 0, \quad j = 1, 2, \cdots, n,\ s \in (0, t].$$

Since the system (16.29) has a solution, we have

$$\sum_{i=1}^m (e_i - r)\beta_{is} = 0, \quad s \in (0, t]$$

and

$$\int_0^t \sum_{i=1}^m (e_i - r)\beta_{is}\mathrm{d}s = 0.$$

This fact implies that

$$\ln(\exp(-rt)Z_t) - \ln Z_0 = 0$$

and

$$\mathcal{M}\{\exp(-rt)Z_t > Z_0\} = 0.$$

That is, the stock model (16.25) is no-arbitrage. Case II: for any given time t and portfolio $(\beta_t, \beta_{1t}, \cdots, \beta_{mt})$, suppose

$$\sum_{j=1}^n \int_0^t \left|\sum_{i=1}^m \sigma_{ij}\beta_{is}\right| \mathrm{d}s \neq 0.$$

Then $\ln(\exp(-rt)Z_t) - \ln Z_0$ is a normal uncertain variable with nonzero variance and

$$\mathcal{M}\{\ln(\exp(-rt)Z_t) - \ln Z_0 \geq 0\} < 1.$$

That is,

$$\mathcal{M}\{\exp(-rt)Z_t \geq Z_0\} < 1$$

and the multifactor stock model (16.25) is no-arbitrage.

Conversely, assume the system (16.29) has no solution. Then there exist real numbers $\alpha_1, \alpha_2, \cdots, \alpha_m$ such that

$$\sum_{i=1}^{m} \sigma_{ij}\alpha_i = 0, \quad j = 1, 2, \cdots, n$$

and

$$\sum_{i=1}^{m} (e_i - r)\alpha_i > 0.$$

Now we take a portfolio

$$(\beta_t, \beta_{1t}, \cdots, \beta_{mt}) \equiv (1 - (\alpha_1 + \alpha_2 + \cdots + \alpha_m), \alpha_1, \alpha_2, \cdots, \alpha_m).$$

Then

$$\ln(\exp(-rt)Z_t) - \ln Z_0 = \int_0^t \sum_{i=1}^{m} (e_i - r)\alpha_i \mathrm{d}s > 0.$$

Thus we have

$$\mathcal{M}\{\exp(-rt)Z_t > Z_0\} = 1.$$

Hence the multifactor stock model (16.25) is arbitrage. The theorem is thus proved.

Theorem 16.8 *The multifactor stock model (16.25) is no-arbitrage if its log-diffusion matrix*

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{m1} & \sigma_{m2} & \cdots & \sigma_{mn} \end{pmatrix} \tag{16.30}$$

has rank m, i.e., the row vectors are linearly independent.

Proof: If the log-diffusion matrix (16.30) has rank m, then the system of equations (16.29) has a solution. It follows from Theorem 16.7 that the multifactor stock model (16.25) is no-arbitrage.

Theorem 16.9 *The multifactor stock model (16.25) is no-arbitrage if its log-drifts are all equal to the interest rate r, i.e.,*

$$e_i = r, \quad i = 1, 2, \cdots, m. \tag{16.31}$$

Proof: Since the log-drifts $e_i = r$ for any $i = 1, 2, \cdots, m$, we immediately have

$$(e_1 - r, e_2 - r, \cdots, e_m - r) \equiv (0, 0, \cdots, 0)$$

that is a linear combination of $(\sigma_{11}, \sigma_{21}, \cdots, \sigma_{m1})$, $(\sigma_{12}, \sigma_{22}, \cdots, \sigma_{m2})$, $\cdots$, $(\sigma_{1n}, \sigma_{2n}, \cdots, \sigma_{mn})$. It follows from Theorem 16.7 that the multifactor stock model (16.25) is no-arbitrage.

16.2 Uncertain Interest Rate Model

Real interest rates do not remain unchanged. Chen and Gao [21] assumed that the interest rate follows an uncertain differential equation and presented an uncertain interest rate model,

$$\mathrm{d}X_t = (m - aX_t)\mathrm{d}t + \sigma\mathrm{d}C_t \tag{16.32}$$

where m, a, σ are positive numbers. Besides, Jiao and Yao [75] investigated the uncertain interest rate model,

$$\mathrm{d}X_t = (m - aX_t)\mathrm{d}t + \sigma\sqrt{X_t}\mathrm{d}C_t. \tag{16.33}$$

More generally, we may assume the interest rate X_t follows a general uncertain differential equation and obtain a general interest rate model,

$$\mathrm{d}X_t = F(t, X_t)\mathrm{d}t + G(t, X_t)\mathrm{d}C_t \tag{16.34}$$

where F and G are two functions, and C_t is a canonical Liu process.

Zero-Coupon Bond

A *zero-coupon bond* is a bond bought at a price lower than its *face value* that is the amount it promises to pay at the maturity date. For simplicity, we assume the face value is always 1 dollar. One problem is how to price a zero-coupon bond.

Definition 16.13 *Let X_t be the uncertain interest rate. Then the price of a zero-coupon bond with a maturity date s is*

$$f = E\left[\exp\left(-\int_0^s X_t\mathrm{d}t\right)\right]. \tag{16.35}$$

Theorem 16.10 *Let X_t^α be the α-path of the uncertain interest rate X_t. Then the price of a zero-coupon bond with maturity date s is*

$$f = \int_0^1 \exp\left(-\int_0^s X_t^\alpha\mathrm{d}t\right)\mathrm{d}\alpha. \tag{16.36}$$

Proof: It follows from Theorem 15.17 that the inverse uncertainty distribution of time integral

$$\int_0^s X_t \mathrm{d}t$$

is

$$\Psi_s^{-1}(\alpha) = \int_0^s X_t^\alpha \mathrm{d}t.$$

Hence the price formula of zero-coupon bond follows from Definition 16.13 immediately.

16.3 Uncertain Currency Model

Liu, Chen and Ralescu [152] assumed that the exchange rate follows an uncertain differential equation and proposed an uncertain currency model,

$$\begin{cases} \mathrm{d}X_t = uX_t\mathrm{d}t \quad \text{(Domestic Currency)} \\ \mathrm{d}Y_t = vY_t\mathrm{d}t \quad \text{(Foreign Currency)} \\ \mathrm{d}Z_t = eZ_t\mathrm{d}t + \sigma Z_t\mathrm{d}C_t \quad \text{(Exchange Rate)} \end{cases} \tag{16.37}$$

where X_t represents the domestic currency with domestic interest rate u, Y_t represents the foreign currency with foreign interest rate v, and Z_t represents the exchange rate that is domestic currency price of one unit of foreign currency at time t. Note that the domestic currency price is $X_t = X_0\exp(ut)$, the foreign currency price is $Y_t = Y_0\exp(vt)$, and the exchange rate is

$$Z_t = Z_0\exp(et + \sigma C_t) \tag{16.38}$$

whose inverse uncertainty distribution is

$$\Phi_t^{-1}(\alpha) = Z_0\exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right). \tag{16.39}$$

European Currency Option

Definition 16.14 *A European currency option is a contract that gives the holder the right to exchange one unit of foreign currency at an expiration time s for K units of domestic currency.*

Suppose that the price of this contract is f in domestic currency. Then the investor pays f for buying the contract at time 0, and receives $(Z_s - K)^+$ in domestic currency at the expiration time s. Thus the expected return of the investor at time 0 is

$$-f + \exp(-us)E[(Z_s - K)^+]. \tag{16.40}$$

On the other hand, the bank receives f for selling the contract at time 0, and pays $(1-K/Z_s)^+$ in foreign currency at the expiration time s. Thus the expected return of the bank at the time 0 is

$$f-\exp(-vs)Z_0E[(1-K/Z_s)^+]. \tag{16.41}$$

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

$$-f+\exp(-us)E[(Z_s-K)^+]=f-\exp(-vs)Z_0E[(1-K/Z_s)^+]. \tag{16.42}$$

Thus the European currency option price is given by the definition below.

Definition 16.15 *(Liu, Chen and Ralescu [152]) Assume a European currency option has a strike price K and an expiration time s. Then the European currency option price is*

$$f=\frac{1}{2}\exp(-us)E[(Z_s-K)^+]+\frac{1}{2}\exp(-vs)Z_0E[(1-K/Z_s)^+]. \tag{16.43}$$

Theorem 16.11 *(Liu, Chen and Ralescu [152]) Assume a European currency option for the uncertain currency model (16.37) has a strike price K and an expiration time s. Then the European currency option price is*

$$\begin{aligned}f=&\frac{1}{2}\exp(-us)\int_0^1\left(Z_0\exp\left(es+\frac{\sigma s\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-K\right)^+\mathrm{d}\alpha\\&+\frac{1}{2}\exp(-vs)\int_0^1\left(Z_0-K/\exp\left(es+\frac{\sigma s\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+\mathrm{d}\alpha.\end{aligned}$$

Proof: Since $(Z_s-K)^+$ and $Z_0(1-K/Z_s)^+$ are increasing functions with respect to Z_s, they have inverse uncertainty distributions

$$\Psi_s^{-1}(\alpha)=\left(Z_0\exp\left(es+\frac{\sigma s\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)-K\right)^+,$$

$$\Upsilon_s^{-1}(\alpha)=\left(Z_0-K/\exp\left(es+\frac{\sigma s\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+,$$

respectively. Thus the European currency option price formula follows from Definition 16.15 immediately.

Remark 16.5: The European currency option price of the uncertain currency model (16.37) is a decreasing function of K, u and v.

Example 16.5: Assume the domestic interest rate $u=0.08$, the foreign interest rate $v=0.07$, the log-drift $e=0.06$, the log-diffusion $\sigma=0.32$, the initial exchange rate $Z_0=5$, the strike price $K=6$ and the expiration time $s=$

2. The Matlab Uncertainty Toolbox (http://orsc.edu.cn/liu/resources.htm) yields the European currency option price

$$f = 0.977.$$

American Currency Option

Definition 16.16 *An American currency option is a contract that gives the holder the right to exchange one unit of foreign currency at any time prior to an expiration time s for K units of domestic currency.*

Suppose that the price of this contract is f in domestic currency. Then the investor pays f for buying the contract, and receives

$$\sup_{0\le t\le s} \exp(-ut)(Z_t - K)^+ \tag{16.44}$$

in domestic currency. Thus the expected return of the investor at time 0 is

$$-f + E\left[\sup_{0\le t\le s} \exp(-ut)(Z_t - K)^+\right]. \tag{16.45}$$

On the other hand, the bank receives f for selling the contract, and pays

$$\sup_{0\le t\le s} \exp(-vt)(1 - K/Z_t)^+. \tag{16.46}$$

in foreign currency. Thus the expected return of the bank at time 0 is

$$f - E\left[\sup_{0\le t\le s} \exp(-vt)Z_0(1 - K/Z_t)^+\right]. \tag{16.47}$$

The fair price of this contract should make the investor and the bank have an identical expected return, i.e.,

$$\begin{aligned} &-f + E\left[\sup_{0\le t\le s} \exp(-ut)(Z_t - K)^+\right] \\ &= f - E\left[\sup_{0\le t\le s} \exp(-vt)Z_0(1 - K/Z_t)^+\right]. \end{aligned} \tag{16.48}$$

Thus the American currency option price is given by the definition below.

Definition 16.17 *(Liu, Chen and Ralescu [152]) Assume an American currency option has a strike price K and an expiration time s. Then the American currency option price is*

$$f = \frac{1}{2}E\left[\sup_{0\le t\le s} \exp(-ut)(Z_t - K)^+\right] + \frac{1}{2}E\left[\sup_{0\le t\le s} \exp(-vt)Z_0(1 - K/Z_t)^+\right].$$

Theorem 16.12 *(Liu, Chen and Ralescu [152]) Assume an American currency option for the uncertain currency model (16.37) has a strike price K and an expiration time s. Then the American currency option price is*

$$f = \frac{1}{2}\int_0^1 \sup_{0\le t\le s} \exp(-ut)\left(Z_0 \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right)^+ \mathrm{d}\alpha$$
$$+\frac{1}{2}\int_0^1 \sup_{0\le t\le s} \exp(-vt)\left(Z_0 - K/\exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+ \mathrm{d}\alpha.$$

Proof: It follows from Theorem 15.13 that $\sup_{0\le t\le s}\exp(-ut)(Z_t - K)^+$ and $\sup_{0\le t\le s}\exp(-vt)Z_0(1-K/Z_t)^+$ have inverse uncertainty distributions

$$\Psi_s^{-1}(\alpha) = \sup_{0\le t\le s} \exp(-ut)\left(Z_0 \exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right) - K\right)^+,$$

$$\Upsilon_s^{-1}(\alpha) = \sup_{0\le t\le s} \exp(-vt)\left(Z_0 - K/\exp\left(et + \frac{\sigma t\sqrt{3}}{\pi}\ln\frac{\alpha}{1-\alpha}\right)\right)^+,$$

respectively. Thus the American currency option price formula follows from Definition 16.17 immediately.

General Currency Model

If the exchange rate follows a general uncertain differential equation, then we have a general currency model,

$$\begin{cases} \mathrm{d}X_t = uX_t\mathrm{d}t \quad \text{(Domestic Currency)} \\ \mathrm{d}Y_t = vY_t\mathrm{d}t \quad \text{(Foreign Currency)} \\ \mathrm{d}Z_t = F(t,Z_t)\mathrm{d}t + G(t,Z_t)\mathrm{d}C_t \quad \text{(Exchange Rate)} \end{cases} \tag{16.49}$$

where u and v are interest rates, F and G are two functions, and C_t is a canonical Liu process.

Note that the α-path Z_t^α of the exchange rate Z_t can be calculated by some numerical methods. Assume the strike price is K and the expiration time is s. It follows from Definition 16.15 and Theorem 15.12 that the European currency option price is

$$f = \frac{1}{2}\int_0^1 \left(\exp(-us)(Z_s^\alpha - K)^+ + \exp(-vs)Z_0(1-K/Z_s^\alpha)^+\right)\mathrm{d}\alpha.$$

It follows from Definition 16.17 and Theorem 15.13 that the American currency option price is

$$f = \frac{1}{2}\int_0^1\left(\sup_{0\le t\le s}\exp(-ut)(Z_t^\alpha - K)^+ + \sup_{0\le t\le s}\exp(-vt)Z_0(1-K/Z_t^\alpha)^+\right)\mathrm{d}\alpha.$$

16.4 Bibliographic Notes

The classical finance theory assumed that stock price, interest rate, and exchange rate follow stochastic differential equations. However, this preassumption was challenged among others by Liu [134] in which a convincing paradox was presented to show why the real stock price is impossible to follow any stochastic differential equations. As an alternative, Liu [134] suggested to develop a theory of uncertain finance.

Uncertain differential equations were first introduced into finance by Liu [125] in 2009 in which an uncertain stock model was proposed and European option price formulas were provided. Besides, Chen [13] derived American option price formulas, Sun and Chen [218] verified Asian option price formulas, and Yao [248] proved a no-arbitrage theorem for this type of uncertain stock model. It is emphasized that other uncertain stock models were also actively investigated by Peng and Yao [182], Yu [259], and Chen, Liu and Ralescu [19], among others.

Uncertain differential equations were used to simulate interest rate by Chen and Gao [21] in 2013 and an uncertain interest rate model was presented. On the basis of this model, the price of zero-coupon bond was also produced. Besides, Jiao and Yao [75] investigated another type of uncertain interest rate model.

Uncertain differential equations were employed to model currency exchange rate by Liu, Chen and Ralescu [152] in which an uncertain currency model was proposed and some currency option price formulas were also derived for the uncertain currency markets. In addition, Shen and Yao [208] discussed another type of uncertain currency model.

Appendix A

Probability Theory

It is generally believed that the study of probability theory was started by Pascal and Fermat in the 17th century when they succeeded in deriving the exact probabilities for certain gambling problem. After that, probability theory was subsequently studied by many researchers. A great progress was achieved when von Mises [226] initialized the concept of sample space in 1931. A complete axiomatic foundation of probability theory was given by Kolmogorov [88] in 1933. Since then, probability theory has been developed steadily and widely applied in science and engineering.

The emphasis in this appendix is mainly on probability measure, random variable, probability distribution, independence, operational law, expected value, variance, moment, entropy, law of large numbers, conditional probability, stochastic process, stochastic calculus, and stochastic differential equation.

A.1 Probability Measure

Let Ω be a nonempty set, and let $\mathcal{A}$ be a σ-algebra over Ω. Each element in $\mathcal{A}$ is called an event. In order to present an axiomatic definition of probability, the following three axioms are assumed:

Axiom 1. *(Normality Axiom)* $\Pr\{\Omega\} = 1$ *for the universal set* Ω.

Axiom 2. *(Nonnegativity Axiom)* $\Pr\{A\} \ge 0$ *for any event* A.

Axiom 3. *(Additivity Axiom) For every countable sequence of mutually disjoint events* $A_1, A_2, \cdots$, *we have*

$$\Pr\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} \Pr\{A_i\}. \tag{A.1}$$

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5

Definition A.1 *The set function* Pr *is called a probability measure if it satisfies the normality, nonnegativity, and additivity axioms.*

Example A.1: Let $\Omega = \{\omega_1, \omega_2, \cdots\}$, and let $\mathcal{A}$ be the power set of Ω. Assume that $p_1, p_2, \cdots$ are nonnegative numbers such that $p_1 + p_2 + \cdots = 1$. Define a set function on $\mathcal{A}$ as

$$\Pr\{A\} = \sum_{\omega_i \in A} p_i. \tag{A.2}$$

Then Pr is a probability measure.

Example A.2: Let ϕ be a nonnegative and integrable function on $\Re$ (the set of real numbers) such that

$$\int_{\Re} \phi(x)\mathrm{d}x = 1. \tag{A.3}$$

Define a set function on the Borel algebra as

$$\Pr\{A\} = \int_A \phi(x)\mathrm{d}x. \tag{A.4}$$

Then Pr is a probability measure.

Definition A.2 *Let* Ω *be a nonempty set, let* $\mathcal{A}$ *be a* σ*-algebra over* Ω*, and let* Pr *be a probability measure. Then the triplet* $(\Omega, \mathcal{A}, \Pr)$ *is called a probability space.*

Example A.3: Let $\Omega = \{\omega_1, \omega_2, \cdots\}$, let $\mathcal{A}$ be the power set of Ω, and let Pr be a probability measure defined by (A.2). Then $(\Omega, \mathcal{A}, \Pr)$ is a probability space.

Example A.4: Let $\Omega = [0, 1]$, let $\mathcal{A}$ be the Borel algebra over Ω, and let Pr be the Lebesgue measure. Then $(\Omega, \mathcal{A}, \Pr)$ is a probability space. For many purposes it is sufficient to use it as the basic probability space.

Theorem A.1 *(Probability Continuity Theorem) Let* $(\Omega, \mathcal{A}, \Pr)$ *be a probability space. If* $A_1, A_2, \cdots \in \mathcal{A}$ *and* $\lim_{i\to\infty} A_i$ *exists, then*

$$\lim_{i\to\infty} \Pr\{A_i\} = \Pr\left\{\lim_{i\to\infty} A_i\right\}. \tag{A.5}$$

Proof: STEP 1: Suppose $\{A_i\}$ is an increasing sequence of events. Write $A_i \to A$ and $A_0 = \emptyset$. Then $\{A_i \backslash A_{i-1}\}$ is a sequence of disjoint events and

$$\bigcup_{i=1}^{\infty} (A_i \backslash A_{i-1}) = A, \quad \bigcup_{i=1}^{k} (A_i \backslash A_{i-1}) = A_k$$

for $k = 1, 2, \cdots$ Thus we have

$$\begin{aligned}\Pr\{A\} &= \Pr\left\{\bigcup_{i=1}^{\infty}(A_i \backslash A_{i-1})\right\} = \sum_{i=1}^{\infty}\Pr\{A_i \backslash A_{i-1}\} \\ &= \lim_{k\to\infty}\sum_{i=1}^{k}\Pr\{A_i \backslash A_{i-1}\} = \lim_{k\to\infty}\Pr\left\{\bigcup_{i=1}^{k}(A_i \backslash A_{i-1})\right\} \\ &= \lim_{k\to\infty}\Pr\{A_k\}.\end{aligned}$$

STEP 2: If $\{A_i\}$ is a decreasing sequence of events, then the sequence $\{A_1 \backslash A_i\}$ is clearly increasing. It follows that

$$\begin{aligned}\Pr\{A_1\} - \Pr\{A\} &= \Pr\left\{\lim_{i\to\infty}(A_1 \backslash A_i)\right\} = \lim_{i\to\infty}\Pr\{A_1 \backslash A_i\} \\ &= \Pr\{A_1\} - \lim_{i\to\infty}\Pr\{A_i\}\end{aligned}$$

which implies that $\Pr\{A_i\} \to \Pr\{A\}$.

STEP 3: If $\{A_i\}$ is a sequence of events such that $A_i \to A$, then for each k, we have

$$\bigcap_{i=k}^{\infty} A_i \subset A_k \subset \bigcup_{i=k}^{\infty} A_i.$$

Since Pr is an increasing set function, we have

$$\Pr\left\{\bigcap_{i=k}^{\infty} A_i\right\} \le \Pr\{A_k\} \le \Pr\left\{\bigcup_{i=k}^{\infty} A_i\right\}.$$

Note that

$$\bigcap_{i=k}^{\infty} A_i \uparrow A, \quad \bigcup_{i=k}^{\infty} A_i \downarrow A.$$

It follows from Steps 1 and 2 that $\Pr\{A_i\} \to \Pr\{A\}$.

Product Probability

Let $(\Omega_k, \mathcal{A}_k, \Pr_k)$, $k = 1, 2, \cdots$ be a sequence of probability spaces. Now we write

$$\Omega = \Omega_1 \times \Omega_2 \times \cdots, \quad \mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \tag{A.6}$$

It has been proved that there is a unique probability measure Pr on the product σ-algebra $\mathcal{A}$ such that

$$\Pr\left\{\prod_{k=1}^{\infty} A_k\right\} = \prod_{k=1}^{\infty}\Pr_k\{A_k\} \tag{A.7}$$

where A_k are arbitrarily chosen events from $\mathcal{A}_k$ for $k = 1, 2, \cdots$, respectively. This conclusion is called *product probability theorem.* Such a probability measure is called *product probability measure*, denoted by

$$\Pr = \Pr_1 \times \Pr_2 \times \cdots \tag{A.8}$$

Remark A.1: Please mention that the product probability theorem cannot be deduced from the three axioms except we presuppose that the product probability meets the three axioms. If I was allowed to reconstruct probability theory, I would like to replace the product probability theorem with Axiom 4: *Let* $(\Omega_k, \mathcal{A}_k, \Pr_k)$ *be probability spaces for* $k = 1, 2, \cdots$ *The product probability measure* $\Pr$ *is a probability measure satisfying*

$$\Pr\left\{\prod_{k=1}^{\infty} A_k\right\} = \prod_{k=1}^{\infty} \Pr_k\{A_k\} \tag{A.9}$$

where A_k *are arbitrarily chosen events from* $\mathcal{A}_k$ *for* $k = 1, 2, \cdots$*, respectively.* One advantage is to force the practitioners to justify the product probability for their own problems.

Definition A.3 *Assume* $(\Omega_k, \mathcal{A}_k, \Pr_k)$ *are probability spaces for* $k = 1, 2, \cdots$ *Let* $\Omega = \Omega_1 \times \Omega_2 \times \cdots$, $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots$ *and* $\Pr = \Pr_1 \times \Pr_2 \times \cdots$ *Then the triplet* $(\Omega, \mathcal{A}, \Pr)$ *is called a product probability space.*

Independence of Events

Definition A.4 *The events* $A_1, A_2, \cdots, A_n$ *are said to be independent if*

$$\Pr\left\{\bigcap_{i=1}^{n} A_i^*\right\} = \prod_{i=1}^{n} \Pr\{A_i^*\}. \tag{A.10}$$

where A_i^* *are arbitrarily chosen from* $\{A_i, \Omega\}$, $i = 1, 2, \cdots, n$*, respectively, and* Ω *is the sure event.*

Remark A.2: Especially, two events A_1 and A_2 are independent if and only if

$$\Pr\{A_1 \cap A_2\} = \Pr\{A_1\} \times \Pr\{A_2\}. \tag{A.11}$$

Example A.5: The impossible event $\emptyset$ is independent of any event A because

$$\Pr\{\emptyset \cap A\} = \Pr\{\emptyset\} = 0 = \Pr\{\emptyset\} \times \Pr\{A\}.$$

Example A.6: The sure event Ω is independent of any event A because

$$\Pr\{\Omega \cap A\} = \Pr\{A\} = \Pr\{\Omega\} \times \Pr\{A\}.$$

Theorem A.2 *Let* $(\Omega_k, \mathcal{A}_k, \Pr_k)$ *be probability spaces and* $A_k \in \mathcal{A}_k$ *for* $k = 1, 2, \cdots, n$. *Then the events*

$$\Omega_1 \times \cdots \times \Omega_{k-1} \times A_k \times \Omega_{k+1} \times \cdots \times \Omega_n, \quad k = 1, 2, \cdots, n \tag{A.12}$$

are always independent in the product probability space. That is, the events

$$A_1, A_2, \cdots, A_n \tag{A.13}$$

are always independent if they are from different probability spaces.

Proof: For simplicity, we only prove the case of $n = 2$. It follows from the product probability theorem that the product probability measure of the intersection is

$$\Pr\{(A_1 \times \Omega_2) \cap (\Omega_1 \times A_2)\} = \Pr\{A_1 \times A_2\} = \Pr_1\{A_1\} \times \Pr_2\{A_2\}.$$

By using $\Pr\{A_1 \times \Omega_2\} = \Pr_1\{A_1\}$ and $\Pr\{\Omega_1 \times A_2\} = \Pr_2\{A_2\}$, we obtain

$$\Pr\{(A_1 \times \Omega_2) \cap (\Omega_1 \times A_2)\} = \Pr\{A_1 \times \Omega_2\} \times \Pr\{\Omega_1 \times A_2\}.$$

Thus $A_1 \times \Omega_2$ and $\Omega_1 \times A_2$ are independent events. Furthermore, since A_1 and A_2 are understood as $A_1 \times \Omega_2$ and $\Omega_1 \times A_2$ in the product probability space, respectively, the two events A_1 and A_2 are also independent.

A.2 Random Variable

Definition A.5 *A random variable is a function from a probability space* $(\Omega, \mathcal{A}, \Pr)$ *to the set of real numbers such that* $\{\xi \in B\}$ *is an event for any Borel set* B.

Example A.7: Take $(\Omega, \mathcal{A}, \Pr)$ to be $\{\omega_1, \omega_2\}$ with $\Pr\{\omega_1\} = \Pr\{\omega_2\} = 0.5$. Then the function

$$\xi(\omega) = \begin{cases} 0, & \text{if } \omega = \omega_1 \\ 1, & \text{if } \omega = \omega_2 \end{cases}$$

is a random variable.

Example A.8: Take $(\Omega, \mathcal{A}, \Pr)$ to be the interval $[0, 1]$ with Borel algebra and Lebesgue measure. We define ξ as an identity function from $[0, 1]$ to $[0, 1]$. Since ξ is a measurable function, it is a random variable.

Definition A.6 *Let* $\xi_1, \xi_2, \cdots, \xi_n$ *be random variables on the probability space* $(\Omega, \mathcal{A}, \Pr)$, *and let* f *be a real-valued measurable function. The*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{A.14}$$

is a random variable defined by

$$\xi(\omega) = f(\xi_1(\omega), \xi_2(\omega), \cdots, \xi_n(\omega)), \quad \forall \omega \in \Omega. \tag{A.15}$$

Theorem A.3 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be random variables, and let f be a real-valued measurable function. Then $f(\xi_1, \xi_2, \cdots, \xi_n)$ is a random variable.*

Proof: Since $\xi_1, \xi_2, \cdots, \xi_n$ are random variables, they are measurable functions from a probability space $(\Omega, \mathcal{A}, \Pr)$ to the set of real numbers. Thus $f(\xi_1, \xi_2, \cdots, \xi_n)$ is also a measurable function from the probability space $(\Omega, \mathcal{A}, \Pr)$ to the set of real numbers. Hence $f(\xi_1, \xi_2, \cdots, \xi_n)$ is a random variable.

A.3 Probability Distribution

Definition A.7 *The probability distribution Φ of a random variable ξ is defined by*

$$\Phi(x) = \Pr\{\xi \le x\} \tag{A.16}$$

for any real number x.

That is, $\Phi(x)$ is the probability that the random variable ξ takes a value less than or equal to x. A function $\Phi : \Re \to [0, 1]$ is a probability distribution if and only if it is an increasing and right-continuous function with

$$\lim_{x \to -\infty} \Phi(x) = 0; \qquad \lim_{x \to +\infty} \Phi(x) = 1. \tag{A.17}$$

Example A.9: Take $(\Omega, \mathcal{A}, \Pr)$ to be $\{\omega_1, \omega_2\}$ with $\Pr\{\omega_1\} = \Pr\{\omega_2\} = 0.5$. We now define a random variable as follows,

$$\xi(\omega) = \begin{cases} 0, & \text{if } \omega = \omega_1 \\ 1, & \text{if } \omega = \omega_2. \end{cases}$$

Then ξ has a probability distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 0 \\ 0.5, & \text{if } 0 \le x < 1 \\ 1, & \text{if } x \ge 1. \end{cases}$$

Definition A.8 *The probability density function ϕ: $\Re \to [0, +\infty)$ of a random variable ξ is a function such that*

$$\Phi(x) = \int_{-\infty}^{x} \phi(y)\mathrm{d}y \tag{A.18}$$

holds for any real number x, where Φ is the probability distribution of the random variable ξ.

Theorem A.4 *(Probability Inversion Theorem) Let ξ be a random variable whose probability density function ϕ exists. Then for any Borel set B, we have*

$$\Pr\{\xi \in B\} = \int_B \phi(y)\mathrm{d}y. \tag{A.19}$$

Proof: Assume that $\mathcal{C}$ is the class of all subsets C of $\Re$ for which the relation

$$\Pr\{\xi \in C\} = \int_C \phi(y)\mathrm{d}y \tag{A.20}$$

holds. We will show that $\mathcal{C}$ contains all Borel sets. On the one hand, we may prove that $\mathcal{C}$ is a monotone class (if $A_i \in \mathcal{C}$ and $A_i \uparrow A$ or $A_i \downarrow A$, then $A \in \mathcal{C}$). On the other hand, we may verify that $\mathcal{C}$ contains all intervals of the form $(-\infty, a]$, $(a, b]$, (b, ∞) and $\emptyset$ since

$$\Pr\{\xi \in (-\infty, a]\} = \Phi(a) = \int_{-\infty}^{a} \phi(y)\mathrm{d}y,$$

$$\Pr\{\xi \in (b, +\infty)\} = \Phi(+\infty) - \Phi(b) = \int_{b}^{+\infty} \phi(y)\mathrm{d}y,$$

$$\Pr\{\xi \in (a, b]\} = \Phi(b) - \Phi(a) = \int_{a}^{b} \phi(y)\mathrm{d}y,$$

$$\Pr\{\xi \in \emptyset\} = 0 = \int_{\emptyset} \phi(y)\mathrm{d}y$$

where Φ is the probability distribution of ξ. Let $\mathcal{F}$ be the algebra consisting of all finite unions of disjoint sets of the form $(-\infty, a]$, $(a, b]$, (b, ∞) and $\emptyset$. Note that for any disjoint sets $C_1, C_2, \cdots, C_m$ of $\mathcal{F}$ and $C = C_1 \cup C_2 \cup \cdots \cup C_m$, we have

$$\Pr\{\xi \in C\} = \sum_{j=1}^{m} \Pr\{\xi \in C_j\} = \sum_{j=1}^{m} \int_{C_j} \phi(y)\mathrm{d}y = \int_C \phi(y)\mathrm{d}y.$$

That is, $C \in \mathcal{C}$. Hence we have $\mathcal{F} \subset \mathcal{C}$. Since the smallest σ-algebra containing $\mathcal{F}$ is just the Borel algebra, the monotone class theorem (if $\mathcal{F} \subset \mathcal{C}$ and $\sigma(\mathcal{F})$ is the smallest σ-algebra containing $\mathcal{F}$, then $\sigma(\mathcal{F}) \subset \mathcal{C}$) implies that $\mathcal{C}$ contains all Borel sets.

Definition A.9 *A random variable ξ has a uniform distribution if its probability density function is*

$$\phi(x) = \frac{1}{b-a}, \quad a \le x \le b \tag{A.21}$$

where a and b are real numbers with $a < b$.

Definition A.10 *A random variable ξ has an exponential distribution if its probability density function is*

$$\phi(x) = \frac{1}{\beta} \exp\left(-\frac{x}{\beta}\right), \quad x \ge 0 \tag{A.22}$$

where β is a positive number.

Definition A.11 *A random variable* ξ *has a normal distribution if its probability density function is*

$$\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty < x < +\infty \tag{A.23}$$

where μ *and* σ *are real numbers with* $\sigma > 0$.

Definition A.12 *A random variable* ξ *has a lognormal distribution if its logarithm is normally distributed, i.e., its probability density function is*

$$\phi(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln x-\mu)^2}{2\sigma^2}\right), \quad x > 0 \tag{A.24}$$

where μ *and* σ *are real numbers with* $\sigma > 0$.

A.4 Independence

Definition A.13 *The random variables* $\xi_1, \xi_2, \cdots, \xi_n$ *are said to be independent if*

$$\Pr\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \prod_{i=1}^{n} \Pr\{\xi_i \in B_i\} \tag{A.25}$$

for any Borel sets $B_1, B_2, \cdots, B_n$.

Example A.10: Let $\xi_1(\omega_1)$ and $\xi_2(\omega_2)$ be random variables on the probability spaces $(\Omega_1, \mathcal{A}_1, \Pr_1)$ and $(\Omega_2, \mathcal{A}_2, \Pr_2)$, respectively. It is clear that they are also random variables on the product probability space $(\Omega_1, \mathcal{A}_1, \Pr_1) \times (\Omega_2, \mathcal{A}_2, \Pr_2)$. Then for any Borel sets B_1 and B_2, we have

$$\begin{aligned}
&\Pr\{(\xi_1 \in B_1) \cap (\xi_2 \in B_2)\} \\
&= \Pr\left\{(\omega_1, \omega_2) \mid \xi_1(\omega_1) \in B_1,\ \xi_2(\omega_2) \in B_2\right\} \\
&= \Pr\left\{(\omega_1 \mid \xi_1(\omega_1) \in B_1) \times (\omega_2 \mid \xi_2(\omega_2) \in B_2)\right\} \\
&= \Pr_1\left\{\omega_1 \mid \xi_1(\omega_1) \in B_1\right\} \times \Pr_2\left\{\omega_2 \mid \xi_2(\omega_2) \in B_2\right\} \\
&= \Pr\left\{\xi_1 \in B_1\right\} \times \Pr\left\{\xi_2 \in B_2\right\}.
\end{aligned}$$

Thus ξ_1 and ξ_2 are independent in the product probability space. In fact, it is true that random variables are always independent if they are defined on different probability spaces.

Theorem A.5 *Let* $\xi_1, \xi_2, \cdots, \xi_n$ *be independent random variables, and let* $f_1, f_2, \cdots, f_n$ *be measurable functions. Then* $f_1(\xi_1), f_2(\xi_2), \cdots, f_n(\xi_n)$ *are independent random variables.*

Proof: For any Borel sets $B_1, B_2, \cdots, B_n$, it follows from the definition of independence that

$$\Pr\left\{\bigcap_{i=1}^{n}(f_i(\xi_i) \in B_i)\right\} = \Pr\left\{\bigcap_{i=1}^{n}(\xi_i \in f_i^{-1}(B_i))\right\}$$

$$= \prod_{i=1}^{n} \Pr\{\xi_i \in f_i^{-1}(B_i)\} = \prod_{i=1}^{n} \Pr\{f_i(\xi_i) \in B_i\}.$$

Thus $f_1(\xi_1), f_2(\xi_2), \cdots, f_n(\xi_n)$ are independent random variables.

A.5 Operational Law

Theorem A.6 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent random variables with probability distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, and let $f : \Re^n \to \Re$ be a measurable function. Then the random variable*

$$\xi = f(\xi_1, \xi_2, \cdots, \xi_n) \tag{A.26}$$

has a probability distribution

$$\Phi(x) = \int_{f(x_1,x_2,\cdots,x_n)\le x} \mathrm{d}\Phi_1(x_1)\mathrm{d}\Phi_2(x_2)\cdots\mathrm{d}\Phi_n(x_n). \tag{A.27}$$

Proof: It follows from the additivity axiom of probability measure and the independence of the random variables $\xi_1, \xi_2, \cdots, \xi_n$ that

$$\begin{aligned}
\Phi(x) &= \Pr\{f(\xi_1, \xi_2, \cdots, \xi_n) \le x\} \\
&= \int_{f(x_1,x_2,\cdots,x_n)\le x} \Pr\left\{\bigcap_{i=1}^{n}(x_i < \xi_i \le x_i + \mathrm{d}x_i)\right\} \\
&= \int_{f(x_1,x_2,\cdots,x_n)\le x} \prod_{i=1}^{n} \Pr\{x_i < \xi_i \le x_i + \mathrm{d}x_i\} \\
&= \int_{f(x_1,x_2,\cdots,x_n)\le x} \prod_{i=1}^{n} (\Phi_i(x_i + \mathrm{d}x_i) - \Phi_i(x_i)) \\
&= \int_{f(x_1,x_2,\cdots,x_n)\le x} \mathrm{d}\Phi_1(x_1)\mathrm{d}\Phi_2(x_2)\cdots\mathrm{d}\Phi_n(x_n).
\end{aligned}$$

The theorem is proved.

Remark A.3: If $\xi_1, \xi_2, \cdots, \xi_n$ have probability density functions $\phi_1, \phi_2, \cdots, \phi_n$, respectively, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has a probability distribution

$$\Phi(x) = \int_{f(x_1,x_2,\cdots,x_n)\le x} \phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n \tag{A.28}$$

because $\mathrm{d}\Phi_i(x_i) = \phi_i(x_i)\mathrm{d}x_i$ for $i = 1, 2, \cdots, n$.

Exercise A.1: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent random variables with probability distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Show that the sum

$$\xi = \xi_1 + \xi_2 + \cdots + \xi_n \tag{A.29}$$

has a probability distribution

$$\Phi(x) = \int_{x_1+x_2+\cdots+x_n \le x} \mathrm{d}\Phi_1(x_1)\mathrm{d}\Phi_2(x_2)\cdots\mathrm{d}\Phi_n(x_n). \tag{A.30}$$

Especially, let ξ_1 and ξ_2 be independent random variables with probability distributions Φ_1 and Φ_2, respectively. Then $\xi = \xi_1 + \xi_2$ has a probability distribution

$$\Phi(x) = \int_{-\infty}^{+\infty} \Phi_1(x - y)\mathrm{d}\Phi_2(y) \tag{A.31}$$

that is called the *convolution* of Φ_1 and Φ_2.

Exercise A.2: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent random variables with probability distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Show that the maximum

$$\xi = \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n \tag{A.32}$$

has a probability distribution

$$\Phi(x) = \Phi_1(x)\Phi_2(x)\cdots\Phi_n(x). \tag{A.33}$$

Exercise A.3: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent random variables with probability distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively. Show that the minimum

$$\xi = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n \tag{A.34}$$

has a probability distribution

$$\Phi(x) = 1 - (1 - \Phi_1(x))(1 - \Phi_2(x))\cdots(1 - \Phi_n(x)). \tag{A.35}$$

Operational Law for Boolean System

Theorem A.7 *Assume that $\xi_1, \xi_2, \cdots, \xi_n$ are independent Boolean random variables, i.e.,*

$$\xi_i = \begin{cases} \text{1 with probability } a_i \\ \text{0 with probability } 1 - a_i \end{cases} \tag{A.36}$$

for $i = 1, 2, \cdots, n$. If f is a Boolean function, then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is a Boolean random variable such that

$$\Pr\{\xi = 1\} = \sum_{(x_1,x_2,\cdots,x_n)\in\{0,1\}^n} \left(\prod_{i=1}^{n} \mu_i(x_i)\right) f(x_1, x_2, \cdots, x_n) \tag{A.37}$$

where

$$\mu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \tag{A.38}$$

for $i = 1, 2, \cdots, n$.

Proof: It follows from the additivity axiom of probability measure and the independence of the random variables $\xi_1, \xi_2, \cdots, \xi_n$ that

$$\begin{aligned} \Pr\{\xi = 1\} &= \sum_{(x_1,x_2,\cdots,x_n)\in\{0,1\}^n} \Pr\left\{\bigcap_{i=1}^n (\xi_i = x_i)\right\} I(f(x_1, x_2, \cdots, x_n) = 1) \\ &= \sum_{(x_1,x_2,\cdots,x_n)\in\{0,1\}^n} \left(\prod_{i=1}^n \Pr\{\xi_i = x_i\}\right) f(x_1, x_2, \cdots, x_n) \\ &= \sum_{(x_1,x_2,\cdots,x_n)\in\{0,1\}^n} \left(\prod_{i=1}^n \mu_i(x_i)\right) f(x_1, x_2, \cdots, x_n) \end{aligned}$$

where $I(\cdot)$ is the indicator function. The theorem is proved.

Exercise A.4: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent Boolean random variables defined by (A.36). Show that

$$\xi = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_n \tag{A.39}$$

is a Boolean random variable such that

$$\Pr\{\xi = 1\} = a_1 a_2 \cdots a_n. \tag{A.40}$$

Exercise A.5: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent Boolean random variables defined by (A.36). Show that

$$\xi = \xi_1 \vee \xi_2 \vee \cdots \vee \xi_n \tag{A.41}$$

is a Boolean random variable such that

$$\Pr\{\xi = 1\} = 1 - (1 - a_1)(1 - a_2) \cdots (1 - a_n). \tag{A.42}$$

Exercise A.6: Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent Boolean random variables defined by (A.36). Show that

$$\xi = k\text{-max}\,[\xi_1, \xi_2, \cdots, \xi_n] \tag{A.43}$$

is a Boolean random variable such that

$$\Pr\{\xi = 1\} = \sum_{(x_1,x_2,\cdots,x_n)\in\{0,1\}^n} \left(\prod_{i=1}^n \mu_i(x_i)\right) k\text{-max}\,[x_1, x_2, \cdots, x_n] \tag{A.44}$$

where

$$\mu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \quad (i = 1, 2, \cdots, n). \tag{A.45}$$

A.6 Expected Value

Definition A.14 *Let ξ be a random variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \Pr\{\xi \ge x\}\mathrm{d}x - \int_{-\infty}^0 \Pr\{\xi \le x\}\mathrm{d}x \tag{A.46}$$

provided that at least one of the two integrals is finite.

Exercise A.7: Assume that ξ is a discrete random variable taking values x_i with probabilities p_i, $i = 1, 2, \cdots, m$, respectively. Show that

$$E[\xi] = \sum_{i=1}^m p_i x_i.$$

Theorem A.8 *Let ξ be a random variable with probability distribution Φ. Then*

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x. \tag{A.47}$$

Proof: It follows from the probability inversion theorem that for almost all numbers x, we have $\Pr\{\xi \ge x\} = 1 - \Phi(x)$ and $\Pr\{\xi \le x\} = \Phi(x)$. By using the definition of expected value operator, we obtain

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} \Pr\{\xi \ge x\}\mathrm{d}x - \int_{-\infty}^0 \Pr\{\xi \le x\}\mathrm{d}x \\ &= \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x. \end{aligned}$$

The theorem is proved.

Theorem A.9 *Let ξ be a random variable with probability distribution Φ. Then*

$$E[\xi] = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x). \tag{A.48}$$

Proof: It follows from the integration by parts and Theorem A.8 that the expected value is

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x \\ &= \int_0^{+\infty} x\mathrm{d}\Phi(x) + \int_{-\infty}^0 x\mathrm{d}\Phi(x) = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x). \end{aligned}$$

The theorem is proved.

Remark A.4: Let $\phi(x)$ be the probability density function of ξ. Then we immediately have

$$E[\xi] = \int_{-\infty}^{+\infty} x\phi(x)\mathrm{d}x \tag{A.49}$$

because $\mathrm{d}\Phi(x) = \phi(x)\mathrm{d}x$.

Theorem A.10 *Let ξ be a random variable with regular probability distribution Φ. Then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha. \tag{A.50}$$

Proof: Substituting $\Phi(x)$ with α and x with $\Phi^{-1}(\alpha)$, it follows from the change of variables of integral and Theorem A.8 that the expected value is

$$E[\xi] = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x) = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha.$$

The theorem is proved.

Theorem A.11 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be independent random variables with probability distributions $\Phi_1, \Phi_2, \cdots, \Phi_n$, respectively, and let $f : \Re^n \to \Re$ be a measurable function. Then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an expected value*

$$E[\xi] = \int_{\Re^n} f(x_1, x_2, \cdots, x_n)\mathrm{d}\Phi_1(x_1)\mathrm{d}\Phi_2(x_2)\cdots\mathrm{d}\Phi_n(x_n). \tag{A.51}$$

Proof: It follows from the operational law of random variables that ξ has a probability distribution

$$\begin{aligned}\Phi(x) &= \int_{f(x_1,x_2,\cdots,x_n)\le x} \mathrm{d}\Phi_1(x_1)\mathrm{d}\Phi_2(x_2)\cdots\mathrm{d}\Phi_n(x_n)\\ &= \int_{\Re^n} I(f(x_1, x_2, \cdots, x_n) \le x)\mathrm{d}\Phi_1(x_1)\mathrm{d}\Phi_2(x_2)\cdots\mathrm{d}\Phi_n(x_n)\end{aligned}$$

where $I(\cdot)$ is the indicator function. Furthermore, we have

$$\mathrm{d}\Phi(x) = \int_{\Re^n} \mathrm{d}I(f(x_1, x_2, \cdots, x_n) \le x)\mathrm{d}\Phi_1(x_1)\mathrm{d}\Phi_2(x_2)\cdots\mathrm{d}\Phi_n(x_n).$$

It follows from Theorem A.9 that

$$\begin{aligned}E[f(\xi)] &= \int_{-\infty}^{+\infty} x \int_{\Re^n} \mathrm{d}I(f(x_1, x_2, \cdots, x_n) \le x)\mathrm{d}\Phi_1(x_1)\mathrm{d}\Phi_2(x_2)\cdots\mathrm{d}\Phi_n(x_n)\\ &= \int_{\Re^n}\int_{-\infty}^{+\infty} x\mathrm{d}I(f(x_1, x_2, \cdots, x_n) \le x)\mathrm{d}\Phi_1(x_1)\mathrm{d}\Phi_2(x_2)\cdots\mathrm{d}\Phi_n(x_n)\\ &= \int_{\Re^n} f(x_1, x_2, \cdots, x_n)\mathrm{d}\Phi_1(x_1)\mathrm{d}\Phi_2(x_2)\cdots\mathrm{d}\Phi_n(x_n).\end{aligned}$$

The theorem is proved.

Theorem A.12 *Let* $\xi_1,\xi_2,\cdots,\xi_n$ *be independent random variables with probability density functions* $\phi_1,\phi_2,\cdots,\phi_n$*, respectively, and let* $f:\Re^n\to\Re$ *be a measurable function. Then* $\xi=f(\xi_1,\xi_2,\cdots,\xi_n)$ *has an expected value*

$$E[\xi]=\int_{\Re^n} f(x_1,x_2,\cdots,x_n)\phi_1(x_1)\phi_2(x_2)\cdots\phi_n(x_n)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n. \quad (A.52)$$

Proof: It follows from $\mathrm{d}\Phi_i(x_i)=\phi_i(x_i)\mathrm{d}x_i$, $i=1,2,\cdots,n$ and Theorem A.11 immediately.

Theorem A.13 *Let* ξ *and* η *be independent random variables with finite expected values. Then*

$$E[\xi\eta]=E[\xi]E[\eta]. \quad (A.53)$$

Proof: Let ξ and η have probability distributions Φ and Ψ, respectively. It follows from Theorem A.11 that

$$\begin{aligned}E[\xi\eta]&=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy\mathrm{d}\Phi(x)\mathrm{d}\Psi(y)\\&=\int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x)\int_{-\infty}^{+\infty} y\mathrm{d}\Psi(y)=E[\xi]E[\eta].\end{aligned}$$

The theorem is verified.

Theorem A.14 *Let* ξ *and* η *be random variables with finite expected values. Then for any numbers* a *and* b*, we have*

$$E[a\xi+b\eta]=aE[\xi]+bE[\eta]. \quad (A.54)$$

Proof: Step 1: We first prove that $E[\xi+b]=E[\xi]+b$ for any real number b. When $b\ge 0$, we have

$$\begin{aligned}E[\xi+b]&=\int_0^{\infty}\Pr\{\xi+b\ge x\}\mathrm{d}x-\int_{-\infty}^0\Pr\{\xi+b\le x\}\mathrm{d}x\\&=\int_0^{\infty}\Pr\{\xi\ge x-b\}\mathrm{d}x-\int_{-\infty}^0\Pr\{\xi\le x-b\}\mathrm{d}x\\&=E[\xi]+\int_0^b\left(\Pr\{\xi\ge x-b\}+\Pr\{\xi<x-b\}\right)\mathrm{d}x\\&=E[\xi]+b.\end{aligned}$$

If $b<0$, then we have

$$E[\xi+b]=E[\xi]-\int_b^0\left(\Pr\{\xi\ge x-b\}+\Pr\{\xi<x-b\}\right)\mathrm{d}x=E[\xi]+b.$$

STEP 2: We prove that $E[a\xi] = aE[\xi]$ for any real number a. If $a = 0$, then the equation $E[a\xi] = aE[\xi]$ holds trivially. If $a > 0$, we have

$$\begin{aligned} E[a\xi] &= \int_0^{\infty} \Pr\{a\xi \geq x\}\mathrm{d}x - \int_{-\infty}^0 \Pr\{a\xi \leq x\}\mathrm{d}x \\ &= \int_0^{\infty} \Pr\left\{\xi \geq \frac{x}{a}\right\}\mathrm{d}x - \int_{-\infty}^0 \Pr\left\{\xi \leq \frac{x}{a}\right\}\mathrm{d}x \\ &= a\int_0^{\infty} \Pr\left\{\xi \geq \frac{x}{a}\right\}\mathrm{d}\left(\frac{x}{a}\right) - a\int_{-\infty}^0 \Pr\left\{\xi \leq \frac{x}{a}\right\}\mathrm{d}\left(\frac{x}{a}\right) \\ &= aE[\xi]. \end{aligned}$$

If $a < 0$, we have

$$\begin{aligned} E[a\xi] &= \int_0^{\infty} \Pr\{a\xi \geq x\}\mathrm{d}x - \int_{-\infty}^0 \Pr\{a\xi \leq x\}\mathrm{d}x \\ &= \int_0^{\infty} \Pr\left\{\xi \leq \frac{x}{a}\right\}\mathrm{d}x - \int_{-\infty}^0 \Pr\left\{\xi \geq \frac{x}{a}\right\}\mathrm{d}x \\ &= a\int_0^{\infty} \Pr\left\{\xi \geq \frac{x}{a}\right\}\mathrm{d}\left(\frac{x}{a}\right) - a\int_{-\infty}^0 \Pr\left\{\xi \leq \frac{x}{a}\right\}\mathrm{d}\left(\frac{x}{a}\right) \\ &= aE[\xi]. \end{aligned}$$

STEP 3: We prove that $E[\xi + \eta] = E[\xi] + E[\eta]$ when both ξ and η are nonnegative simple random variables taking values $a_1, a_2, \cdots, a_m$ and $b_1, b_2, \cdots, b_n$, respectively. Then $\xi + \eta$ is also a nonnegative simple random variable taking values $a_i + b_j$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$. Thus we have

$$\begin{aligned} E[\xi + \eta] &= \sum_{i=1}^{m}\sum_{j=1}^{n}(a_i + b_j)\Pr\{\xi = a_i, \eta = b_j\} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_i \Pr\{\xi = a_i, \eta = b_j\} + \sum_{i=1}^{m}\sum_{j=1}^{n} b_j \Pr\{\xi = a_i, \eta = b_j\} \\ &= \sum_{i=1}^{m} a_i \Pr\{\xi = a_i\} + \sum_{j=1}^{n} b_j \Pr\{\eta = b_j\} \\ &= E[\xi] + E[\eta]. \end{aligned}$$

STEP 4: We prove that $E[\xi + \eta] = E[\xi] + E[\eta]$ when both ξ and η are nonnegative random variables. For every $i \geq 1$ and every $\omega \in \Omega$, we define

$$\xi_i(\omega) = \begin{cases} \dfrac{k-1}{2^i}, & \text{if } \dfrac{k-1}{2^i} \leq \xi(\omega) < \dfrac{k}{2^i},\ k = 1, 2, \cdots, i2^i \\ i, & \text{if } i \leq \xi(\omega), \end{cases}$$

$$\eta_i(\omega) = \begin{cases} \dfrac{k-1}{2^i}, & \text{if } \dfrac{k-1}{2^i} \le \eta(\omega) < \dfrac{k}{2^i},\ k = 1, 2, \cdots, i2^i \\ i, & \text{if } i \le \eta(\omega). \end{cases}$$

Then $\{\xi_i\}$, $\{\eta_i\}$ and $\{\xi_i + \eta_i\}$ are three sequences of nonnegative simple random variables such that $\xi_i \uparrow \xi$, $\eta_i \uparrow \eta$ and $\xi_i + \eta_i \uparrow \xi + \eta$ as $i \to \infty$. Note that the functions $\Pr\{\xi_i > x\}$, $\Pr\{\eta_i > x\}$, $\Pr\{\xi_i + \eta_i > x\}$, $i = 1, 2, \cdots$ are also simple. It follows from the probability continuity theorem that

$$\Pr\{\xi_i > x\} \uparrow \Pr\{\xi > x\},\ \forall x \ge 0$$

as $i \to \infty$. Since the expected value $E[\xi]$ exists, we have

$$E[\xi_i] = \int_0^{+\infty} \Pr\{\xi_i > x\}\mathrm{d}x \to \int_0^{+\infty} \Pr\{\xi > x\}\mathrm{d}x = E[\xi]$$

as $i \to \infty$. Similarly, we may prove that $E[\eta_i] \to E[\eta]$ and $E[\xi_i+\eta_i] \to E[\xi+\eta]$ as $i \to \infty$. It follows from Step 3 that $E[\xi + \eta] = E[\xi] + E[\eta]$.

Step 5: We prove that $E[\xi+\eta] = E[\xi]+E[\eta]$ when ξ and η are arbitrary random variables. Define

$$\xi_i(\omega) = \begin{cases} \xi(\omega), & \text{if } \xi(\omega) \ge -i \\ -i, & \text{otherwise,} \end{cases} \qquad \eta_i(\omega) = \begin{cases} \eta(\omega), & \text{if } \eta(\omega) \ge -i \\ -i, & \text{otherwise.} \end{cases}$$

Since the expected values $E[\xi]$ and $E[\eta]$ are finite, we have

$$\lim_{i\to\infty} E[\xi_i] = E[\xi], \quad \lim_{i\to\infty} E[\eta_i] = E[\eta], \quad \lim_{i\to\infty} E[\xi_i + \eta_i] = E[\xi + \eta].$$

Note that $(\xi_i + i)$ and $(\eta_i + i)$ are nonnegative random variables. It follows from Steps 1 and 4 that

$$\begin{aligned} E[\xi + \eta] &= \lim_{i\to\infty} E[\xi_i + \eta_i] \\ &= \lim_{i\to\infty} (E[(\xi_i + i) + (\eta_i + i)] - 2i) \\ &= \lim_{i\to\infty} (E[\xi_i + i] + E[\eta_i + i] - 2i) \\ &= \lim_{i\to\infty} (E[\xi_i] + i + E[\eta_i] + i - 2i) \\ &= \lim_{i\to\infty} E[\xi_i] + \lim_{i\to\infty} E[\eta_i] \\ &= E[\xi] + E[\eta]. \end{aligned}$$

Step 6: The linearity $E[a\xi + b\eta] = aE[\xi] + bE[\eta]$ follows immediately from Steps 2 and 5. The theorem is proved.

Theorem A.15 *Let ξ be a random variable, and let t be a positive number. If $E[|\xi|^t] < \infty$, then*

$$\lim_{x\to\infty} x^t \Pr\{|\xi| \ge x\} = 0. \tag{A.55}$$

Conversely, let ξ be a random variable satisfying (A.55) for some $t > 0$. Then $E[|\xi|^s] < \infty$ for any $0 \le s < t$.

Proof: It follows from the definition of expected value that

$$E[|\xi|^t] = \int_0^{\infty} \Pr\{|\xi|^t \geq r\}\mathrm{d}r < \infty.$$

Thus we have

$$\lim_{x\to\infty} \int_{x^t/2}^{\infty} \Pr\{|\xi|^t \geq r\}\mathrm{d}r = 0.$$

The equation (A.55) is proved by the following relation,

$$\int_{x^t/2}^{\infty} \Pr\{|\xi|^t \geq r\}\mathrm{d}r \geq \int_{x^t/2}^{x^t} \Pr\{|\xi|^t \geq r\}\mathrm{d}r \geq \frac{1}{2}x^t \Pr\{|\xi| \geq x\}.$$

Conversely, if (A.55) holds, then there exists a number a such that

$$x^t \Pr\{|\xi| \geq x\} \leq 1, \quad \forall x \geq a.$$

Thus we have

$$\begin{aligned} E[|\xi|^s] &= \int_0^{a} \Pr\left\{|\xi|^s \geq r\right\}\mathrm{d}r + \int_a^{+\infty} \Pr\left\{|\xi|^s \geq r\right\}\mathrm{d}r \\ &\leq \int_0^{a} \Pr\left\{|\xi|^s \geq r\right\}\mathrm{d}r + \int_0^{+\infty} sr^{s-1} \Pr\left\{|\xi| \geq r\right\}\mathrm{d}r \\ &\leq \int_0^{a} \Pr\left\{|\xi|^s \geq r\right\}\mathrm{d}r + s\int_0^{+\infty} r^{s-t-1}\mathrm{d}r \\ &< +\infty. \quad \left(\text{by } \int_0^{\infty} r^p \mathrm{d}r < \infty \text{ for any } p < -1\right) \end{aligned}$$

The theorem is proved.

Example A.11: The condition (A.55) does not ensure that $E[|\xi|^t] < \infty$. We consider the positive random variable

$$\xi = \sqrt[t]{\frac{2^i}{i}} \text{ with probability } \frac{1}{2^i}, \quad i = 1, 2, \cdots$$

It is clear that

$$\lim_{x\to\infty} x^t \Pr\{\xi \geq x\} = \lim_{n\to\infty} \left(\sqrt[t]{\frac{2^n}{n}}\right)^t \sum_{i=n}^{\infty} \frac{1}{2^i} = \lim_{n\to\infty} \frac{2}{n} = 0.$$

However, the expected value of ξ^t is

$$E[\xi^t] = \sum_{i=1}^{\infty} \left(\sqrt[t]{\frac{2^i}{i}}\right)^t \cdot \frac{1}{2^i} = \sum_{i=1}^{\infty} \frac{1}{i} = \infty.$$

Theorem A.16 *Let ξ be a random variable, and let f be a nonnegative function. If f is even and increasing on $[0,\infty)$, then for any given number $t>0$, we have*

$$\Pr\{|\xi|\ge t\}\le\frac{E[f(\xi)]}{f(t)}. \tag{A.56}$$

Proof: It is clear that $\Pr\{|\xi|\ge f^{-1}(r)\}$ is a monotone decreasing function of r on $[0,\infty)$. It follows from the nonnegativity of $f(\xi)$ that

$$\begin{aligned}
E[f(\xi)] &= \int_0^{+\infty}\Pr\{f(\xi)\ge x\}\mathrm{d}x=\int_0^{+\infty}\Pr\{|\xi|\ge f^{-1}(x)\}\mathrm{d}x\\
&\ge \int_0^{f(t)}\Pr\{|\xi|\ge f^{-1}(x)\}\mathrm{d}x\ge\int_0^{f(t)}\Pr\{|\xi|\ge f^{-1}(f(t))\}\mathrm{d}x\\
&= \int_0^{f(t)}\Pr\{|\xi|\ge t\}\mathrm{d}x=f(t)\cdot\Pr\{|\xi|\ge t\}
\end{aligned}$$

which proves the inequality.

Theorem A.17 *(Markov Inequality) Let ξ be a random variable. Then for any given numbers $t>0$ and $p>0$, we have*

$$\Pr\{|\xi|\ge t\}\le\frac{E[|\xi|^p]}{t^p}. \tag{A.57}$$

Proof: It is a special case of Theorem A.16 when $f(x)=|x|^p$.

A.7 Variance

Definition A.15 *Let ξ be a random variable with finite expected value e. Then the variance of ξ is defined by $V[\xi]=E[(\xi-e)^2]$.*

Since $(\xi-e)^2$ is a nonnegative random variable, we also have

$$V[\xi]=\int_0^{+\infty}\Pr\{(\xi-e)^2\ge x\}\mathrm{d}x. \tag{A.58}$$

Theorem A.18 *If ξ is a random variable whose variance exists, and a and b are real numbers, then $V[a\xi+b]=a^2V[\xi]$.*

Proof: Let e be the expected value of ξ. Then $E[a\xi+b]=ae+b$. It follows from the definition of variance that

$$V[a\xi+b]=E\left[(a\xi+b-ae-b)^2\right]=a^2E[(\xi-e)^2]=a^2V[\xi].$$

Theorem A.19 *Let ξ be a random variable with expected value e. Then $V[\xi]=0$ if and only if $\Pr\{\xi=e\}=1$. That is, the random variable ξ is essentially the constant e.*

Proof: We first assume $V[\xi]=0$. It follows from the equation (A.58) that

$$\int_0^{+\infty}\Pr\{(\xi-e)^2\ge x\}\mathrm{d}x=0$$

which implies $\Pr\{(\xi-e)^2\ge x\}=0$ for any $x>0$. Hence we have

$$\Pr\{(\xi-e)^2=0\}=1.$$

That is, $\Pr\{\xi=e\}=1$. Conversely, assume $\Pr\{\xi=e\}=1$. Then we immediately have $\Pr\{(\xi-e)^2=0\}=1$ and $\Pr\{(\xi-e)^2\ge x\}=0$ for any $x>0$. Thus

$$V[\xi]=\int_0^{+\infty}\Pr\{(\xi-e)^2\ge x\}\mathrm{d}x=0.$$

The theorem is proved.

Theorem A.20 *If $\xi_1,\xi_2,\cdots,\xi_n$ are independent random variables with finite variances, then*

$$V[\xi_1+\xi_2+\cdots+\xi_n]=V[\xi_1]+V[\xi_2]+\cdots+V[\xi_n]. \tag{A.59}$$

Proof: Let $\xi_1,\xi_2,\cdots,\xi_n$ have expected values $e_1,e_2,\cdots,e_n$, respectively. Then we have

$$E[\xi_1+\xi_2+\cdots+\xi_n]=e_1+e_2+\cdots+e_n.$$

It follows from the definition of variance that

$$V\left[\sum_{i=1}^n\xi_i\right]=\sum_{i=1}^n E\left[(\xi_i-e_i)^2\right]+2\sum_{i=1}^{n-1}\sum_{j=i+1}^n E\left[(\xi_i-e_i)(\xi_j-e_j)\right].$$

Since $\xi_1,\xi_2,\cdots,\xi_n$ are independent, $E\left[(\xi_i-e_i)(\xi_j-e_j)\right]=0$ for all i,j with $i\ne j$. Thus (A.59) holds.

Theorem A.21 *(Chebyshev Inequality) Let ξ be a random variable whose variance exists. Then for any given number $t>0$, we have*

$$\Pr\left\{|\xi-E[\xi]|\ge t\right\}\le\frac{V[\xi]}{t^2}. \tag{A.60}$$

Proof: It is a special case of Theorem A.16 when the random variable ξ is replaced with $\xi-E[\xi]$, and $f(x)=x^2$.

Theorem A.22 *(Kolmogorov Inequality) Let $\xi_1,\xi_2,\cdots,\xi_n$ be independent random variables with finite expected values. Write $S_i=\xi_1+\xi_2+\cdots+\xi_i$ for each $i\ge 1$. Then for any given number $t>0$, we have*

$$\Pr\left\{\max_{1\le i\le n}|S_i-E[S_i]|\ge t\right\}\le\frac{V[S_n]}{t^2}. \tag{A.61}$$

Proof: Without loss of generality, assume that $E[\xi_i]=0$ for each i. We set

$$A_1=\{|S_1|\ge t\},\quad A_i=\{|S_j|<t, j=1,2,\cdots,i-1,\ \text{and}\ |S_i|\ge t\}$$

for $i=2,3,\cdots,n$. It is clear that

$$A=\left\{\max_{1\le i\le n}|S_i|\ge t\right\}$$

is the union of disjoint sets $A_1,A_2,\cdots,A_n$. Since $E[S_n]=0$, we have

$$V[S_n]=\int_0^{+\infty}\Pr\{S_n^2\ge r\}\mathrm{d}r\ge\sum_{k=1}^n\int_0^{+\infty}\Pr\left\{(S_n^2\ge r)\cap A_k\right\}\mathrm{d}r. \tag{A.62}$$

Now for any k with $1\le k\le n$, it follows from the independence that

$$\begin{aligned}
&\int_0^{+\infty}\Pr\left\{(S_n^2\ge r)\cap A_k\right\}\mathrm{d}r\\
=&\int_0^{+\infty}\Pr\left\{((S_k+\xi_{k+1}+\cdots+\xi_n)^2\ge r)\cap A_k\right\}\mathrm{d}r\\
=&\int_0^{+\infty}\Pr\left\{(S_k^2+\xi_{k+1}^2+\cdots+\xi_n^2\ge r)\cap A_k\right\}\mathrm{d}r\\
&\quad+2\sum_{j=k+1}^n E[I_{A_k}S_k]E[\xi_j]+\sum_{j\ne l;j,l=k+1}^n\Pr\{A_k\}E[\xi_j]E[\xi_l]\\
\ge&\int_0^{+\infty}\Pr\left\{(S_k^2\ge r)\cap A_k\right\}\mathrm{d}r\\
\ge&\ t^2\Pr\{A_k\}.
\end{aligned}$$

Using (A.62), we get

$$V[S_n]\ge t^2\sum_{i=1}^n\Pr\{A_i\}=t^2\Pr\{A\}$$

which implies that the Kolmogorov inequality holds.

Theorem A.23 *Let ξ be a random variable with probability distribution Φ and expected value e. Then*

$$V[\xi]=\int_0^{+\infty}(1-\Phi(e+\sqrt{x})+\Phi(e-\sqrt{x}))\mathrm{d}x. \tag{A.63}$$

Proof: It follows from the additivity of probability measure that the variance is

$$\begin{aligned} V[\xi] &= \int_0^{+\infty} \Pr\{(\xi - e)^2 \geq x\}\mathrm{d}x \\ &= \int_0^{+\infty} \Pr\{(\xi \geq e + \sqrt{x}) \cup (\xi \leq e - \sqrt{x})\}\mathrm{d}x \\ &= \int_0^{+\infty} (\Pr\{\xi \geq e + \sqrt{x}\} + \Pr\{\xi \leq e - \sqrt{x}\})\mathrm{d}x \\ &= \int_0^{+\infty} (1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x}))\mathrm{d}x. \end{aligned}$$

The theorem is proved.

Theorem A.24 *Let ξ be a random variable with probability distribution Φ and expected value e. Then*

$$V[\xi] = \int_{-\infty}^{+\infty} (x - e)^2 \mathrm{d}\Phi(x). \tag{A.64}$$

Proof: For the equation (A.63), substituting $e + \sqrt{y}$ with x and y with $(x - e)^2$, the change of variables and integration by parts produce

$$\int_0^{+\infty} (1 - \Phi(e + \sqrt{y}))\mathrm{d}y = \int_e^{+\infty} (1 - \Phi(x))\mathrm{d}(x - e)^2 = \int_e^{+\infty} (x - e)^2 \mathrm{d}\Phi(x).$$

Similarly, substituting $e - \sqrt{y}$ with x and y with $(x - e)^2$, we obtain

$$\int_0^{+\infty} \Phi(e - \sqrt{y})\mathrm{d}y = \int_e^{-\infty} \Phi(x)\mathrm{d}(x - e)^2 = \int_{-\infty}^{e} (x - e)^2 \mathrm{d}\Phi(x).$$

It follows that the variance is

$$V[\xi] = \int_e^{+\infty} (x - e)^2 \mathrm{d}\Phi(x) + \int_{-\infty}^{e} (x - e)^2 \mathrm{d}\Phi(x) = \int_{-\infty}^{+\infty} (x - e)^2 \mathrm{d}\Phi(x).$$

The theorem is verified.

Remark A.5: Let $\phi(x)$ be the probability density function of ξ. Then we immediately have

$$V[\xi] = \int_{-\infty}^{+\infty} (x - e)^2 \phi(x)\mathrm{d}x. \tag{A.65}$$

because $\mathrm{d}\Phi(x) = \phi(x)\mathrm{d}x$.

Theorem A.25 *Let ξ be a random variable with regular probability distribution Φ and expected value e. Then*

$$V[\xi] = \int_0^1 (\Phi^{-1}(\alpha) - e)^2 \mathrm{d}\alpha. \tag{A.66}$$

Proof: Substituting $\Phi(x)$ with α and x with $\Phi^{-1}(\alpha)$, it follows from the change of variables of integral and Theorem A.24 that the variance is

$$V[\xi] = \int_{-\infty}^{+\infty} (x - e)^2 \mathrm{d}\Phi(x) = \int_0^1 (\Phi^{-1}(\alpha) - e)^2 \mathrm{d}\alpha.$$

The theorem is verified.

A.8 Moment

Definition A.16 *Let ξ be a random variable, and let k be a positive integer. Then $E[\xi^k]$ is called the kth moment of ξ.*

Theorem A.26 *Let ξ be a random variable with probability distribution Φ, and let k be an odd number. Then the k-th moment of ξ is*

$$E[\xi^k] = \int_0^{+\infty} (1 - \Phi(\sqrt[k]{x}))\mathrm{d}x - \int_{-\infty}^0 \Phi(\sqrt[k]{x})\mathrm{d}x. \tag{A.67}$$

Proof: Since k is an odd number, it follows from the definition of expected value operator that

$$\begin{aligned} E[\xi^k] &= \int_0^{+\infty} \Pr\{\xi^k \geq x\}\mathrm{d}x - \int_{-\infty}^0 \Pr\{\xi^k \leq x\}\mathrm{d}x \\ &= \int_0^{+\infty} \Pr\{\xi \geq \sqrt[k]{x}\}\mathrm{d}x - \int_{-\infty}^0 \Pr\{\xi \leq \sqrt[k]{x}\}\mathrm{d}x \\ &= \int_0^{+\infty} (1 - \Phi(\sqrt[k]{x}))\mathrm{d}x - \int_{-\infty}^0 \Phi(\sqrt[k]{x})\mathrm{d}x. \end{aligned}$$

The theorem is proved.

Theorem A.27 *Let ξ be a random variable with probability distribution Φ, and let k be an even number. Then the k-th moment of ξ is*

$$E[\xi^k] = \int_0^{+\infty} (1 - \Phi(\sqrt[k]{x}) + \Phi(-\sqrt[k]{x}))\mathrm{d}x. \tag{A.68}$$

Proof: Since k is an odd number, ξ^k is a nonnegative random variable. It follows from the definition of expected value operator that

$$\begin{aligned}E[\xi^k] &= \int_0^{+\infty} \Pr\{\xi^k \geq x\}\mathrm{d}x \\ &= \int_0^{+\infty} \Pr\{(\xi \geq \sqrt[k]{x}) \cup (\xi \leq -\sqrt[k]{x})\}\mathrm{d}x \\ &= \int_0^{+\infty} (\Pr\{\xi \geq \sqrt[k]{x}\} + \Pr\{\xi \leq -\sqrt[k]{x}\})\mathrm{d}x \\ &= \int_0^{+\infty} (1 - \Phi(\sqrt[k]{x}) + \Phi(-\sqrt[k]{x}))\mathrm{d}x.\end{aligned}$$

The theorem is verified.

Theorem A.28 *Let ξ be a random variable with probability distribution Φ, and let k be a positive integer. Then the k-th moment of ξ is*

$$E[\xi^k] = \int_{-\infty}^{+\infty} x^k \mathrm{d}\Phi(x). \tag{A.69}$$

Proof: When k is an odd number, Theorem A.26 says that the k-th moment is

$$E[\xi^k] = \int_0^{+\infty} (1 - \Phi(\sqrt[k]{y}))\mathrm{d}y - \int_{-\infty}^0 \Phi(\sqrt[k]{y})\mathrm{d}y.$$

Substituting $\sqrt[k]{y}$ with x and y with x^k, the change of variables and integration by parts produce

$$\int_0^{+\infty} (1 - \Phi(\sqrt[k]{y}))\mathrm{d}y = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x^k = \int_0^{+\infty} x^k \mathrm{d}\Phi(x)$$

and

$$\int_{-\infty}^0 \Phi(\sqrt[k]{y})\mathrm{d}y = \int_{-\infty}^0 \Phi(x)\mathrm{d}x^k = -\int_{-\infty}^0 x^k \mathrm{d}\Phi(x).$$

Thus we have

$$E[\xi^k] = \int_0^{+\infty} x^k \mathrm{d}\Phi(x) + \int_{-\infty}^0 x^k \mathrm{d}\Phi(x) = \int_{-\infty}^{+\infty} x^k \mathrm{d}\Phi(x).$$

When k is an even number, Theorem A.27 says that the k-th moment is

$$E[\xi^k] = \int_0^{+\infty} (1 - \Phi(\sqrt[k]{y}) + \Phi(-\sqrt[k]{y}))\mathrm{d}y.$$

Substituting $\sqrt[k]{y}$ with x and y with x^k, the change of variables and integration by parts produce

$$\int_0^{+\infty} (1 - \Phi(\sqrt[k]{y}))\mathrm{d}y = \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x^k = \int_0^{+\infty} x^k \mathrm{d}\Phi(x).$$

Similarly, substituting $-\sqrt[k]{y}$ with x and y with x^k, we obtain

$$\int_0^{+\infty} \Phi(-\sqrt[k]{y})\mathrm{d}y = \int_{-\infty}^0 \Phi(x)\mathrm{d}x^k = \int_{-\infty}^0 x^k \mathrm{d}\Phi(x).$$

It follows that the k-th moment is

$$E[\xi^k] = \int_0^{+\infty} x^k \mathrm{d}\Phi(x) + \int_{-\infty}^0 x^k \mathrm{d}\Phi(x) = \int_{-\infty}^{+\infty} x^k \mathrm{d}\Phi(x).$$

The theorem is thus verified for any positive integer k.

Theorem A.29 *Let ξ be a random variable with regular probability distribution Φ, and let k be a positive integer. Then the k-th moment of ξ is*

$$E[\xi^k] = \int_0^1 (\Phi^{-1}(\alpha))^k \mathrm{d}\alpha. \tag{A.70}$$

Proof: Substituting $\Phi(x)$ with α and x with $\Phi^{-1}(\alpha)$, it follows from the change of variables of integral and Theorem A.28 that the k-th moment is

$$E[\xi^k] = \int_{-\infty}^{+\infty} x^k \mathrm{d}\Phi(x) = \int_0^1 (\Phi^{-1}(\alpha))^k \mathrm{d}\alpha.$$

The theorem is verified.

A.9 Entropy

Given a random variable, what is the degree of difficulty of predicting the specified value that the random variable will take? In order to answer this question, Shannon [205] defined a concept of entropy as a measure of uncertainty.

Definition A.17 *Let ξ be a random variable with probability density function ϕ. Then its entropy is defined by*

$$H[\xi] = -\int_{-\infty}^{+\infty} \phi(x) \ln \phi(x)\mathrm{d}x. \tag{A.71}$$

Example A.12: Let ξ be a uniformly distributed random variable on $[a, b]$. Then its entropy is $H[\xi] = \ln(b-a)$. This example shows that the entropy may assume both positive and negative values since $\ln(b-a) < 0$ if $b-a < 1$; and $\ln(b-a) > 0$ if $b-a > 1$.

Example A.13: Let ξ be an exponentially distributed random variable with expected value β. Then its entropy is $H[\xi] = 1 + \ln \beta$.

Example A.14: Let ξ be a normally distributed random variable with expected value e and variance σ^2. Then its entropy is $H[\xi] = 1/2 + \ln \sqrt{2\pi}\sigma$.

Maximum Entropy Principle

Given some constraints, for example, expected value and variance, there are usually multiple compatible probability distributions. For this case, we would like to select the distribution that maximizes the value of entropy and satisfies the prescribed constraints. This method is often referred to as the *maximum entropy principle* (Jaynes [70]).

Example A.15: Let ξ be a random variable on $[a, b]$ whose probability density function exists. The maximum entropy principle attempts to find the probability density function $\phi(x)$ that maximizes the entropy

$$-\int_a^b \phi(x)\ln\phi(x)\mathrm{d}x$$

subject to the natural constraint $\int_a^b \phi(x)\mathrm{d}x = 1$. The Lagrangian is

$$L = -\int_a^b \phi(x)\ln\phi(x)\mathrm{d}x - \lambda\left(\int_a^b \phi(x)\mathrm{d}x - 1\right).$$

It follows from the Euler-Lagrange equation that the maximum entropy probability density function meets

$$\ln\phi(x) + 1 + \lambda = 0$$

and has the form $\phi(x) = \exp(-1-\lambda)$. Substituting it into the natural constraint, we get

$$\phi^*(x) = \frac{1}{b-a}, \quad a \le x \le b$$

which is just a uniform probability density function, and the maximum entropy is $H[\xi^*] = \ln(b-a)$.

Example A.16: Let ξ be a random variable on $(-\infty, +\infty)$ whose probability density function exists. Assume that the expected value and variance of ξ are prescribed to be μ and σ^2, respectively. The maximum entropy probability density function $\phi(x)$ should maximize the entropy

$$-\int_{-\infty}^{+\infty} \phi(x)\ln\phi(x)\mathrm{d}x$$

subject to the constraints

$$\int_{-\infty}^{+\infty} \phi(x)\mathrm{d}x = 1, \quad \int_{-\infty}^{+\infty} x\phi(x)\mathrm{d}x = \mu, \quad \int_{-\infty}^{+\infty} (x-\mu)^2\phi(x)\mathrm{d}x = \sigma^2.$$

The Lagrangian is

$$L = -\int_{-\infty}^{+\infty} \phi(x) \ln \phi(x) \mathrm{d}x - \lambda_1 \left(\int_{-\infty}^{+\infty} \phi(x) \mathrm{d}x - 1 \right)$$
$$-\lambda_2 \left(\int_{-\infty}^{+\infty} x\phi(x) \mathrm{d}x - \mu \right) - \lambda_3 \left(\int_{-\infty}^{+\infty} (x-\mu)^2 \phi(x) \mathrm{d}x - \sigma^2 \right).$$

The maximum entropy probability density function meets Euler-Lagrange equation

$$\ln \phi(x) + 1 + \lambda_1 + \lambda_2 x + \lambda_3 (x-\mu)^2 = 0$$

and has the form $\phi(x) = \exp(-1 - \lambda_1 - \lambda_2 x - \lambda_3 (x-\mu)^2)$. Substituting it into the constraints, we get

$$\phi^*(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right), \quad x \in \Re$$

which is just a normal probability density function, and the maximum entropy is $H[\xi^*] = 1/2 + \ln \sqrt{2\pi}\sigma$.

A.10 Random Sequence

Random sequence is a sequence of random variables indexed by integers. This section introduces four convergence concepts of random sequence: convergence almost surely (a.s.), convergence in probability, convergence in mean, and convergence in distribution.

Table A.1: Relations among Convergence Concepts

Convergence Almost Surely	$\searrow$	Convergence in Probability	$\rightarrow$	Convergence in Distribution
Convergence in Mean	$\nearrow$			

Definition A.18 *The random sequence $\{\xi_i\}$ is said to be convergent a.s. to ξ if and only if there exists an event A with $\Pr\{A\} = 1$ such that*

$$\lim_{i \to \infty} |\xi_i(\omega) - \xi(\omega)| = 0 \tag{A.72}$$

for every $\omega \in A$. In that case we write $\xi_i \to \xi$, a.s.

Definition A.19 *The random sequence* $\{\xi_i\}$ *is said to be convergent in probability to* ξ *if*

$$\lim_{i\to\infty} \Pr\{|\xi_i - \xi| \geq \varepsilon\} = 0 \tag{A.73}$$

for every $\varepsilon > 0$.

Definition A.20 *The random sequence* $\{\xi_i\}$ *is said to be convergent in mean to* ξ *if*

$$\lim_{i\to\infty} E[|\xi_i - \xi|] = 0. \tag{A.74}$$

Definition A.21 *Let* $\Phi, \Phi_1, \Phi_2, \cdots$ *be the probability distributions of random variables* $\xi, \xi_1, \xi_2, \cdots$, *respectively. We say the random sequence* $\{\xi_i\}$ *converges in distribution to* ξ *if*

$$\lim_{i\to\infty} \Phi_i(x) = \Phi(x) \tag{A.75}$$

for all x *at which* $\Phi(x)$ *is continuous.*

Convergence Almost Surely vs. Convergence in Probability

Theorem A.30 *The random sequence* $\{\xi_i\}$ *converges a.s. to* ξ *if and only if for every* $\varepsilon > 0$, *we have*

$$\lim_{n\to\infty} \Pr\left\{\bigcup_{i=n}^{\infty}\{|\xi_i - \xi| \geq \varepsilon\}\right\} = 0. \tag{A.76}$$

Proof: For every $i \geq 1$ and $\varepsilon > 0$, we define

$$A = \left\{\omega \in \Omega \;\Big|\; \lim_{i\to\infty} \xi_i(\omega) \neq \xi(\omega)\right\},$$
$$A_i(\varepsilon) = \left\{\omega \in \Omega \;\Big|\; |\xi_i(\omega) - \xi(\omega)| \geq \varepsilon\right\}.$$

It is clear that

$$A = \bigcup_{\varepsilon>0}\left(\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} A_i(\varepsilon)\right).$$

Note that $\xi_i \to \xi$, a.s. if and only if $\Pr\{A\} = 0$. That is, $\xi_i \to \xi$, a.s. if and only if

$$\Pr\left\{\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} A_i(\varepsilon)\right\} = 0$$

for every $\varepsilon > 0$. Since

$$\bigcup_{i=n}^{\infty} A_i(\varepsilon) \downarrow \bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty} A_i(\varepsilon),$$

it follows from the probability continuity theorem that

$$\lim_{n\to\infty}\Pr\left\{\bigcup_{i=n}^{\infty}A_i(\varepsilon)\right\}=\Pr\left\{\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty}A_i(\varepsilon)\right\}=0.$$

The theorem is proved.

Theorem A.31 *If the random sequence $\{\xi_i\}$ converges a.s. to ξ, then $\{\xi_i\}$ converges in probability to ξ.*

Proof: It follows from the convergence a.s. and Theorem A.30 that

$$\lim_{n\to\infty}\Pr\left\{\bigcup_{i=n}^{\infty}\{|\xi_i-\xi|\ge\varepsilon\}\right\}=0$$

for each $\varepsilon>0$. For every $n\ge 1$, since

$$\{|\xi_n-\xi|\ge\varepsilon\}\subset\bigcup_{i=n}^{\infty}\{|\xi_i-\xi|\ge\varepsilon\},$$

we have $\Pr\{|\xi_n-\xi|\ge\varepsilon\}\to 0$ as $n\to\infty$. Hence the theorem holds.

Example A.17: Convergence in probability does not imply convergence a.s. For example, take $(\Omega,\mathcal{A},\Pr)$ to be the interval $[0,1]$ with Borel algebra and Lebesgue measure. For any positive integer i, there is an integer j such that $i=2^j+k$, where k is an integer between 0 and 2^j-1. We define a random variable by

$$\xi_i(\omega)=\begin{cases}1, & \text{if } k/2^j\le\omega\le(k+1)/2^j\\ 0, & \text{otherwise}\end{cases}$$

for $i=1,2,\cdots$ and $\xi=0$. For any small number $\varepsilon>0$, we have

$$\Pr\{|\xi_i-\xi|\ge\varepsilon\}=\frac{1}{2^j}\to 0$$

as $i\to\infty$. That is, the sequence $\{\xi_i\}$ converges in probability to ξ. However, for any $\omega\in[0,1]$, there is an infinite number of intervals of the form $[k/2^j,(k+1)/2^j]$ containing ω. Thus $\xi_i(\omega)\not\to 0$ as $i\to\infty$. In other words, the sequence $\{\xi_i\}$ does not converge a.s. to ξ.

Convergence in Probability vs. Convergence in Mean

Theorem A.32 *If the random sequence $\{\xi_i\}$ converges in mean to ξ, then $\{\xi_i\}$ converges in probability to ξ.*

Proof: It follows from the Markov inequality that, for any given number $\varepsilon > 0$,

$$\Pr\{|\xi_i - \xi| \ge \varepsilon\} \le \frac{E[|\xi_i - \xi|]}{\varepsilon} \to 0$$

as $i \to \infty$. Thus $\{\xi_i\}$ converges in probability to ξ.

Example A.18: Convergence in probability does not imply convergence in mean. For example, take $(\Omega, \mathcal{A}, \Pr)$ to be $\{\omega_1, \omega_2, \cdots\}$ with $\Pr\{\omega_j\} = 1/2^j$ for $j = 1, 2, \cdots$ The random variables are defined by

$$\xi_i(\omega_j) = \begin{cases} 2^i, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi = 0$. For any small number $\varepsilon > 0$, we have

$$\Pr\{|\xi_i - \xi| \ge \varepsilon\} = \frac{1}{2^i} \to 0$$

as $i \to \infty$. That is, the sequence $\{\xi_i\}$ converges in probability to ξ. However, we have

$$E[|\xi_i - \xi|] = 2^i \cdot \frac{1}{2^i} = 1$$

for each i. That is, the sequence $\{\xi_i\}$ does not converge in mean to ξ.

Convergence Almost Surely vs. Convergence in Mean

Example A.19: Convergence a.s. does not imply convergence in mean. For example, take $(\Omega, \mathcal{A}, \Pr)$ to be $\{\omega_1, \omega_2, \cdots\}$ with $\Pr\{\omega_j\} = 1/2^j$ for $j = 1, 2, \cdots$ The random variables are defined by

$$\xi_i(\omega_j) = \begin{cases} 2^i, & \text{if } j = i \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi = 0$. Then $\{\xi_i\}$ converges a.s. to ξ. However, the sequence $\{\xi_i\}$ does not converge in mean to ξ.

Example A.20: Convergence in mean does not imply convergence a.s. For example, take $(\Omega, \mathcal{A}, \Pr)$ to be the interval $[0, 1]$ with Borel algebra and Lebesgue measure. For any positive integer i, there is an integer j such that $i = 2^j + k$, where k is an integer between 0 and $2^j - 1$. We define a random variable by

$$\xi_i(\omega) = \begin{cases} 1, & \text{if } k/2^j \le \omega \le (k+1)/2^j \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ and $\xi = 0$. Then

$$E[|\xi_i - \xi|] = \frac{1}{2^j} \to 0$$

as $i \to \infty$. That is, the sequence $\{\xi_i\}$ converges in mean to ξ. However, $\{\xi_i\}$ does not converge a.s. to ξ.

Convergence in Probability vs. Convergence in Distribution

Theorem A.33 *If the random sequence $\{\xi_i\}$ converges in probability to ξ, then $\{\xi_i\}$ converges in distribution to ξ.*

Proof: Let x be any given continuity point of the probability distribution Φ. On the one hand, for any $y > x$, we have

$$\{\xi_i \le x\} = \{\xi_i \le x, \xi \le y\} \cup \{\xi_i \le x, \xi > y\} \subset \{\xi \le y\} \cup \{|\xi_i - \xi| \ge y - x\}$$

which implies that

$$\Phi_i(x) \le \Phi(y) + \Pr\{|\xi_i - \xi| \ge y - x\}.$$

Since $\{\xi_i\}$ converges in probability to ξ, we have $\Pr\{|\xi_i - \xi| \ge y - x\} \to 0$. Thus we obtain $\limsup_{i\to\infty} \Phi_i(x) \le \Phi(y)$ for any $y > x$. Letting $y \to x$, we get

$$\limsup_{i\to\infty} \Phi_i(x) \le \Phi(x). \tag{A.77}$$

On the other hand, for any $z < x$, we have

$$\{\xi \le z\} = \{\xi \le z, \xi_i \le x\} \cup \{\xi \le z, \xi_i > x\} \subset \{\xi_i \le x\} \cup \{|\xi_i - \xi| \ge x - z\}$$

which implies that

$$\Phi(z) \le \Phi_i(x) + \Pr\{|\xi_i - \xi| \ge x - z\}.$$

Since $\Pr\{|\xi_i - \xi| \ge x - z\} \to 0$ as $i \to \infty$, we obtain $\Phi(z) \le \liminf_{i\to\infty} \Phi_i(x)$ for any $z < x$. Letting $z \to x$, we get

$$\Phi(x) \le \liminf_{i\to\infty} \Phi_i(x). \tag{A.78}$$

It follows from (A.77) and (A.78) that $\Phi_i(x) \to \Phi(x)$ as $i \to \infty$. The theorem is proved.

Example A.21: Convergence in distribution does not imply convergence in probability. For example, take $(\Omega, \mathcal{A}, \Pr)$ to be $\{\omega_1, \omega_2\}$ with $\Pr\{\omega_1\} = \Pr\{\omega_2\} = 0.5$, and

$$\xi(\omega) = \begin{cases} -1, & \text{if } \omega = \omega_1 \\ 1, & \text{if } \omega = \omega_2. \end{cases}$$

We also define $\xi_i = -\xi$ for all i. Then ξ_i and ξ are identically distributed. Thus $\{\xi_i\}$ converges in distribution to ξ. But, for any small number $\varepsilon > 0$, we have $\Pr\{|\xi_i - \xi| > \varepsilon\} = \Pr\{\Omega\} = 1$. That is, the sequence $\{\xi_i\}$ does not converge in probability to ξ.

A.11 Law of Large Numbers

The laws of large numbers include two types: (a) the weak laws of large numbers dealing with convergence in probability; (b) the strong laws of large numbers dealing with convergence a.s. In order to introduce them, we will denote

$$S_n = \xi_1 + \xi_2 + \cdots + \xi_n \tag{A.79}$$

for each n throughout this section.

Weak Laws of Large Numbers

Theorem A.34 *(Chebyshev's Weak Law of Large Numbers) Let $\xi_1, \xi_2, \cdots$ be a sequence of independent but not necessarily identically distributed random variables with finite expected values. If there exists a number $a > 0$ such that $V[\xi_i] < a$ for all i, then $(S_n - E[S_n])/n$ converges in probability to 0 as $n \to \infty$.*

Proof: For any given $\varepsilon > 0$, it follows from Chebyshev inequality that

$$\Pr\left\{\left|\frac{S_n - E[S_n]}{n}\right| \ge \varepsilon\right\} \le \frac{1}{\varepsilon^2}V\left[\frac{S_n}{n}\right] = \frac{V[S_n]}{\varepsilon^2 n^2} \le \frac{a}{\varepsilon^2 n} \to 0$$

as $n \to \infty$. The theorem is proved. Especially, if those random variables have a common expected value e, then S_n/n converges in probability to e.

Theorem A.35 *Let $\xi_1, \xi_2, \cdots$ be a sequence of iid random variables with finite expected value e. Then S_n/n converges in probability to e as $n \to \infty$.*

Proof: For each i, since the expected value of ξ_i is finite, there exists $\beta > 0$ such that $E[|\xi_i|] < \beta < \infty$. Let α be an arbitrary positive number, and let n be an arbitrary positive integer. We define

$$\xi_i^* = \begin{cases} \xi_i, & \text{if } |\xi_i| < n\alpha \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, 2, \cdots$ It is clear that $\{\xi_i^*\}$ is a sequence of iid random variables. Let e_n^* be the common expected value of ξ_i^*, and $S_n^* = \xi_1^* + \xi_2^* + \cdots + \xi_n^*$. Then we have

$$V[\xi_i^*] \le E[\xi_i^{*2}] \le n\alpha E[|\xi_i^*|] \le n\alpha\beta,$$

$$E\left[\frac{S_n^*}{n}\right] = \frac{E[\xi_1^*] + E[\xi_2^*] + \cdots + E[\xi_n^*]}{n} = e_n^*,$$

$$V\left[\frac{S_n^*}{n}\right] = \frac{V[\xi_1^*] + V[\xi_2^*] + \cdots + V[\xi_n^*]}{n^2} \le \alpha\beta.$$

It follows from Chebyshev inequality that

$$\Pr\left\{\left|\frac{S_n^*}{n} - e_n^*\right| \ge \varepsilon\right\} \le \frac{1}{\varepsilon^2} V\left[\frac{S_n^*}{n}\right] \le \frac{\alpha\beta}{\varepsilon^2} \tag{A.80}$$

for every $\varepsilon > 0$. It is also clear that $e_n^* \to e$ as $n \to \infty$ by Lebesgue dominated convergence theorem. Thus there exists an integer N^* such that $|e_n^* - e| < \varepsilon$ whenever $n \ge N^*$. Applying (A.80), we get

$$\Pr\left\{\left|\frac{S_n^*}{n} - e\right| \ge 2\varepsilon\right\} \le \Pr\left\{\left|\frac{S_n^*}{n} - e_n^*\right| \ge \varepsilon\right\} \le \frac{\alpha\beta}{\varepsilon^2} \tag{A.81}$$

for any $n \ge N^*$. It follows from the iid hypothesis and Theorem A.15 that

$$\Pr\{S_n^* \ne S_n\} \le \sum_{i=1}^{n} \Pr\{|\xi_i| \ge n\alpha\} \le n \Pr\{|\xi_1| \ge n\alpha\} \to 0$$

as $n \to \infty$. Thus there exists an integer N^{**} such that

$$\Pr\{S_n^* \ne S_n\} \le \alpha, \quad \forall n \ge N^{**}.$$

Applying (A.81), for all $n \ge N^* \vee N^{**}$, we have

$$\Pr\left\{\left|\frac{S_n}{n} - e\right| \ge 2\varepsilon\right\} \le \frac{\alpha\beta}{\varepsilon^2} + \alpha \to 0$$

as $\alpha \to 0$. It follows that S_n/n converges in probability to e as $n \to \infty$.

Strong Laws of Large Numbers

Lemma A.1 *(Toeplitz Lemma) Let $a, a_1, a_2, \cdots$ be a sequence of real numbers such that $a_i \to a$ as $i \to \infty$. Then*

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = a. \tag{A.82}$$

Proof: Let $\varepsilon > 0$ be given. Since $a_i \to a$, there exists an integer N such that

$$|a_i - a| < \frac{\varepsilon}{2}, \quad \forall i \ge N.$$

It is also able to choose an integer $N^* > N$ such that

$$\frac{1}{N^*} \sum_{i=1}^{N} |a_i - a| < \frac{\varepsilon}{2}.$$

Thus for any $n > N^*$, we have

$$\left|\frac{1}{n}\sum_{i=1}^{n} a_i - a\right| \le \frac{1}{N^*}\sum_{i=1}^{N} |a_i - a| + \frac{1}{n}\sum_{i=N+1}^{n} |a_i - a| < \varepsilon.$$

It follows from the arbitrariness of ε that Toeplitz lemma holds.

Lemma A.2 *(Kronecker Lemma) Let $a_1, a_2, \cdots$ be a sequence of real numbers such that $\sum_{i=1}^{\infty} a_i$ converges. Then*

$$\lim_{n\to\infty} \frac{a_1 + 2a_2 + \cdots + na_n}{n} = 0. \tag{A.83}$$

Proof: We set $s_0 = 0$ and $s_i = a_1 + a_2 + \cdots + a_i$ for $i = 1, 2, \cdots$ Then we have

$$\frac{1}{n}\sum_{i=1}^{n} ia_i = \frac{1}{n}\sum_{i=1}^{n} i(s_i - s_{i-1}) = s_n - \frac{1}{n}\sum_{i=1}^{n-1} s_i.$$

The sequence $\{s_i\}$ converges to a finite limit, say s. It follows from Toeplitz lemma that $\sum_{i=1}^{n-1} s_i/n \to s$ as $n \to \infty$. Thus Kronecker lemma is proved.

Theorem A.36 *(Kolmogorov Strong Law of Large Numbers) Let $\xi_1, \xi_2, \cdots$ be a sequence of independent random variables with finite expected values. If*

$$\sum_{i=1}^{\infty} \frac{V[\xi_i]}{i^2} < \infty, \tag{A.84}$$

then

$$\frac{S_n - E[S_n]}{n} \to 0, \ \ a.s. \tag{A.85}$$

as $n \to \infty$.

Proof: Since $\xi_1, \xi_2, \cdots$ are independent random variables with finite expected values, for every given $\varepsilon > 0$, we have

$$\begin{aligned}
&\Pr\left\{\bigcup_{j=0}^{\infty}\left(\left|\sum_{i=n}^{n+j}\frac{\xi_i - E[\xi_i]}{i}\right| \ge \varepsilon\right)\right\} \\
&= \lim_{m\to\infty} \Pr\left\{\bigcup_{j=0}^{m}\left(\left|\sum_{i=n}^{n+j}\frac{\xi_i}{i} - E\left[\sum_{i=n}^{n+j}\frac{\xi_i}{i}\right]\right| \ge \varepsilon\right)\right\} \\
&= \lim_{m\to\infty} \Pr\left\{\max_{0\le j\le m}\left|\sum_{i=n}^{n+j}\frac{\xi_i}{i} - E\left[\sum_{i=n}^{n+j}\frac{\xi_i}{i}\right]\right| \ge \varepsilon\right\} \\
&\le \lim_{m\to\infty} \frac{1}{\varepsilon^2} V\left[\sum_{i=n}^{n+m}\frac{\xi_i}{i}\right] \quad \text{(by Kolmogorov inequality)} \\
&= \lim_{m\to\infty} \frac{1}{\varepsilon^2}\sum_{i=n}^{n+m}\frac{V[\xi_i]}{i^2} = \frac{1}{\varepsilon^2}\sum_{i=n}^{\infty}\frac{V[\xi_i]}{i^2} \to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus $\sum_{i=1}^{\infty}(\xi_i - E[\xi_i])/i$ converges a.s. Applying Kronecker lemma, we obtain

$$\frac{S_n - E[S_n]}{n} = \frac{1}{n}\sum_{i=1}^{n} i\left(\frac{\xi_i - E[\xi_i]}{i}\right) \to 0, \quad \text{a.s.}$$

as $n \to \infty$. The theorem is proved.

Theorem A.37 *(Strong Law of Large Numbers) Let $\xi_1, \xi_2, \cdots$ be a sequence of iid random variables with finite expected value e. Then*

$$\frac{S_n}{n} \to e, \ \ a.s. \tag{A.86}$$

as $n \to \infty$.

Proof: For each $i \geq 1$, let ξ_i^* be ξ_i truncated at i, i.e.,

$$\xi_i^* = \begin{cases} \xi_i, & \text{if } |\xi_i| < i \\ 0, & \text{otherwise,} \end{cases}$$

and write $S_n^* = \xi_1^* + \xi_2^* + \cdots + \xi_n^*$. Then we have

$$V[\xi_i^*] \leq E[\xi_i^{*2}] \leq \sum_{j=1}^{i} j^2 \Pr\{j-1 \leq |\xi_1| < j\}$$

for all i. Thus

$$\begin{aligned}
\sum_{i=1}^{\infty} \frac{V[\xi_i^*]}{i^2} &\leq \sum_{i=1}^{\infty}\sum_{j=1}^{i} \frac{j^2}{i^2} \Pr\{j-1 \leq |\xi_1| < j\} \\
&= \sum_{j=1}^{\infty} j^2 \Pr\{j-1 \leq |\xi_1| < j\} \sum_{i=j}^{\infty} \frac{1}{i^2} \\
&\leq 2\sum_{j=1}^{\infty} j \Pr\{j-1 \leq |\xi_1| < j\} \quad \text{by } \sum_{i=j}^{\infty}\frac{1}{i^2} \leq \frac{2}{j} \\
&= 2 + 2\sum_{j=1}^{\infty} (j-1) \Pr\{j-1 \leq |\xi_1| < j\} \\
&\leq 2 + 2e < \infty.
\end{aligned}$$

It follows from Theorem A.36 that

$$\frac{S_n^* - E[S_n^*]}{n} \to 0, \quad \text{a.s.} \tag{A.87}$$

as $n \to \infty$. Note that $\xi_i^* \uparrow \xi_i$ as $i \to \infty$. Using the Lebesgue dominated convergence theorem, we conclude that $E[\xi_i^*] \to e$. It follows from Toeplitz Lemma that

$$\frac{E[S_n^*]}{n} = \frac{E[\xi_1^*] + E[\xi_2^*] + \cdots + E[\xi_n^*]}{n} \to e. \tag{A.88}$$

Since $(\xi_i - \xi_i^*) \to 0$, a.s. as $i \to \infty$, Toeplitz Lemma states that

$$\frac{S_n - S_n^*}{n} = \frac{1}{n}\sum_{i=1}^{n} (\xi_i - \xi_i^*) \to 0, \quad \text{a.s.} \tag{A.89}$$

It follows from (A.87), (A.88) and (A.89) that $S_n/n \to e$ a.s. as $n \to \infty$.

A.12 Conditional Probability

We consider the probability of an event A after it has been learned that some other event B has occurred. This new probability is called the *conditional probability* of A given B.

Definition A.22 *Let $(\Omega, \mathcal{A}, \Pr)$ be a probability space, and $A, B \in \mathcal{A}$. Then the conditional probability of A given B is defined by*

$$\Pr\{A|B\} = \frac{\Pr\{A \cap B\}}{\Pr\{B\}} \tag{A.90}$$

provided that $\Pr\{B\} > 0$.

Example A.22: Let ξ be an exponentially distributed random variable with expected value β. Then for any real numbers $a > 0$ and $x > 0$, the conditional probability of $\xi \geq a + x$ given $\xi \geq a$ is

$$\Pr\{\xi \geq a + x|\xi \geq a\} = \exp(-x/\beta) = \Pr\{\xi \geq x\}$$

which means that the conditional probability is identical to the original probability. This is the so-called *memoryless property* of exponential distribution. In other words, it is as good as new if it is functioning on inspection.

Theorem A.38 *(Bayes Formula) Let the events $A_1, A_2, \cdots, A_n$ form a partition of the space Ω such that $\Pr\{A_i\} > 0$ for $i = 1, 2, \cdots, n$, and let B be an event with $\Pr\{B\} > 0$. Then we have*

$$\Pr\{A_k|B\} = \frac{\Pr\{A_k\}\Pr\{B|A_k\}}{\sum_{i=1}^{n} \Pr\{A_i\}\Pr\{B|A_i\}} \tag{A.91}$$

for $k = 1, 2, \cdots, n$.

Proof: Since $A_1, A_2, \cdots, A_n$ form a partition of the space Ω, we immediately have

$$\Pr\{B\} = \sum_{i=1}^{n} \Pr\{A_i \cap B\} = \sum_{i=1}^{n} \Pr\{A_i\}\Pr\{B|A_i\}$$

which is also called the *formula for total probability*. Thus, for any k, we have

$$\Pr\{A_k|B\} = \frac{\Pr\{A_k \cap B\}}{\Pr\{B\}} = \frac{\Pr\{A_k\}\Pr\{B|A_k\}}{\sum_{i=1}^{n} \Pr\{A_i\}\Pr\{B|A_i\}}.$$

The theorem is proved.

Remark A.6: Especially, let A and B be two events with $\Pr\{A\} > 0$ and $\Pr\{B\} > 0$. Then A and A^c form a partition of the space Ω, and the *Bayes formula* is

$$\Pr\{A|B\} = \frac{\Pr\{A\}\Pr\{B|A\}}{\Pr\{B\}}. \tag{A.92}$$

Remark A.7: In statistical applications, the events $A_1, A_2, \cdots, A_n$ are often called *hypotheses*. Furthermore, for each i, the $\Pr\{A_i\}$ is called *a priori* probability of A_i, and $\Pr\{A_i|B\}$ is called *a posteriori* probability of A_i after the occurrence of event B.

Definition A.23 *The conditional probability distribution* Φ*:* $\Re \to [0,1]$ *of a random variable* ξ *given* B *is defined by*

$$\Phi(x|B) = \Pr\{\xi \le x|B\} \tag{A.93}$$

provided that $\Pr\{B\} > 0$*.*

Example A.23: Let ξ and η be random variables. Then the conditional probability distribution of ξ given $\eta = y$ is

$$\Phi(x|\eta = y) = \Pr\{\xi \le x|\eta = y\} = \frac{\Pr\{\xi \le x, \eta = y\}}{\Pr\{\eta = y\}}$$

provided that $\Pr\{\eta = y\} > 0$.

Definition A.24 *The conditional probability density function* ϕ *of a random variable* ξ *given* B *is a nonnegative function such that*

$$\Phi(x|B) = \int_{-\infty}^{x} \phi(y|B)\mathrm{d}y, \quad \forall x \in \Re \tag{A.94}$$

where $\Phi(x|B)$ *is the conditional probability distribution of* ξ *given* B*.*

Example A.24: Let (ξ, η) be a random vector with joint probability density function ψ. Then the marginal probability density functions of ξ and η are

$$f(x) = \int_{-\infty}^{+\infty} \psi(x,y)\mathrm{d}y, \quad g(y) = \int_{-\infty}^{+\infty} \psi(x,y)\mathrm{d}x,$$

respectively. Furthermore, we have

$$\Pr\{\xi \le x, \eta \le y\} = \int_{-\infty}^{x}\int_{-\infty}^{y} \psi(r,t)\mathrm{d}r\mathrm{d}t = \int_{-\infty}^{y}\left[\int_{-\infty}^{x} \frac{\psi(r,t)}{g(t)}\mathrm{d}r\right] g(t)\mathrm{d}t$$

which implies that the conditional probability distribution of ξ given $\eta = y$ is

$$\Phi(x|\eta = y) = \int_{-\infty}^{x} \frac{\psi(r,y)}{g(y)}\mathrm{d}r, \quad \text{a.s.} \tag{A.95}$$

and the conditional probability density function of ξ given $\eta = y$ is

$$\phi(x|\eta = y) = \frac{\psi(x,y)}{g(y)} = \frac{\psi(x,y)}{\displaystyle\int_{-\infty}^{+\infty} \psi(x,y)\mathrm{d}x}, \quad \text{a.s.} \tag{A.96}$$

Note that (A.95) and (A.96) are defined only for $g(y) \neq 0$. In fact, the set $\{y|g(y) = 0\}$ has probability 0. Especially, if ξ and η are independent random variables, then $\psi(x,y) = f(x)g(y)$ and $\phi(x|\eta = y) = f(x)$.

A.13 Random Set

It is believed that the earliest study of random set was Robbins [198] in 1944, and a rigorous definition was given by Matheron [167] in 1975. In this book, let us redefine the concept of random set and propose a concept of membership function for it.

Definition A.25 *A random set is a function ξ from a probability space $(\Omega, \mathcal{A}, \Pr)$ to a collection of sets such that both $\{B \subset \xi\}$ and $\{\xi \subset B\}$ are events for any Borel set B.*

Example A.25: Take a probability space $(\Omega, \mathcal{A}, \Pr)$ to be $\{\omega_1, \omega_2, \omega_3\}$. Then the set-valued function

$$\xi(\omega) = \begin{cases} [1,3], & \text{if } \omega = \omega_1 \\ [2,4], & \text{if } \omega = \omega_2 \\ [3,5], & \text{if } \omega = \omega_3 \end{cases} \tag{A.97}$$

is a random set on $(\Omega, \mathcal{A}, \Pr)$.

Definition A.26 *A random set ξ is said to have a membership function μ if for any Borel set B, we have*

$$\Pr\{B \subset \xi\} = \inf_{x \in B} \mu(x), \tag{A.98}$$

$$\Pr\{\xi \subset B\} = 1 - \sup_{x \in B^c} \mu(x). \tag{A.99}$$

The above equations will be called probability inversion formulas.

Remark A.8: When a random set ξ does have a membership function μ, we immediately have

$$\mu(x) = \Pr\{x \in \xi\}. \tag{A.100}$$

Example A.26: A crisp set A of real numbers is a special random set $\xi(\omega) \equiv A$. Show that such a random set has a membership function

$$\mu(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases} \tag{A.101}$$

that is just the characteristic function of A.

Example A.27: Take a probability space $(\Omega, \mathcal{A}, \Pr)$ to be the interval $[0, 1]$ with Borel algebra and Lebesgue measure. Then the random set

$$\xi(\omega) = \left[-\sqrt{1-\omega},\ \sqrt{1-\omega}\right] \tag{A.102}$$

has a membership function

$$\mu(x) = \begin{cases} 1 - x^2, & \text{if } x \in [-1, 1] \\ 0, & \text{otherwise.} \end{cases} \tag{A.103}$$

Theorem A.39 *A real-valued function μ is a membership function if and only if*

$$0 \le \mu(x) \le 1. \tag{A.104}$$

Proof: If μ is a membership function of some random set ξ, then $\mu(x) = \Pr\{x \in \xi\}$ and $0 \le \mu(x) \le 1$. Conversely, suppose μ is a function such that $0 \le \mu(x) \le 1$. Take a probability space $(\Omega, \mathcal{A}, \Pr)$ to be the interval $[0, 1]$ with Borel algebra and Lebesgue measure. Then the random set

$$\xi(\omega) = \{x \mid \mu(x) \ge \omega\} \tag{A.105}$$

has the membership function μ.

Theorem A.40 *Let ξ be a random set with membership function μ. Then its complement ξ^c has a membership function*

$$\lambda(x) = 1 - \mu(x). \tag{A.106}$$

Proof: In order to prove $1-\mu$ is a membership function of ξ^c, we must verify the two probability inversion formulas. Let B be a Borel set. It follows from the definition of membership function that

$$\Pr\{B \subset \xi^c\} = \Pr\{\xi \subset B^c\} = 1 - \sup_{x \in (B^c)^c} \mu(x) = \inf_{x \in B} (1 - \mu(x)),$$

$$\Pr\{\xi^c \subset B\} = \Pr\{B^c \subset \xi\} = \inf_{x \in B^c} \mu(x) = 1 - \sup_{x \in B^c} (1 - \mu(x)).$$

Thus ξ^c has a membership function $1 - \mu$.

Definition A.27 *Let ξ be a random set with membership function μ. Then the set-valued function*

$$\mu^{-1}(\alpha) = \left\{x \in \Re \;\middle|\; \mu(x) \ge \alpha\right\}, \quad \forall\alpha \in [0,1] \tag{A.107}$$

is called the inverse membership function of ξ. Sometimes, for each given α, the set $\mu^{-1}(\alpha)$ is also called the α-cut of μ.

Theorem A.41 *(Sufficient and Necessary Condition) A function $\mu^{-1}(\alpha)$ is an inverse membership function if and only if it is a monotone decreasing set-valued function with respect to $\alpha \in [0,1]$. That is,*

$$\mu^{-1}(\alpha) \subset \mu^{-1}(\beta), \quad \text{if } \alpha > \beta. \tag{A.108}$$

Proof: Suppose $\mu^{-1}(\alpha)$ is an inverse membership function of some random set. For any $x \in \mu^{-1}(\alpha)$, we have $\mu(x) \ge \alpha$. Since $\alpha > \beta$, we have $\mu(x) > \beta$ and then $x \in \mu^{-1}(\beta)$. Hence $\mu^{-1}(\alpha) \subset \mu^{-1}(\beta)$. Conversely, suppose $\mu^{-1}(\alpha)$ is a monotone decreasing set-valued function. Then

$$\mu(x) = \sup\left\{\alpha \in [0,1] \;\middle|\; x \in \mu^{-1}(\alpha)\right\}$$

is a membership function of some random set. It is easy to verify that $\mu^{-1}(\alpha)$ is the inverse membership function of the random set. The theorem is proved.

Theorem A.42 *Let ξ be a random set with inverse membership function $\mu^{-1}(\alpha)$. Then for each $\alpha \in [0,1]$, we have*

$$\Pr\{\mu^{-1}(\alpha) \subset \xi\} \ge \alpha, \tag{A.109}$$

$$\Pr\{\xi \subset \mu^{-1}(\alpha)\} \ge 1 - \alpha. \tag{A.110}$$

Proof: For each $x \in \mu^{-1}(\alpha)$, we have $\mu(x) \ge \alpha$. It follows from the probability inversion formula that

$$\Pr\{\mu^{-1}(\alpha) \subset \xi\} = \inf_{x \in \mu^{-1}(\alpha)} \mu(x) \ge \alpha.$$

For each $x \notin \mu^{-1}(\alpha)$, we have $\mu(x) < \alpha$. It follows from the probability inversion formula that

$$\Pr\{\xi \subset \mu^{-1}(\alpha)\} = 1 - \sup_{x \notin \mu^{-1}(\alpha)} \mu(x) \ge 1 - \alpha.$$

A.14 Stochastic Process

A stochastic process is essentially a sequence of random variables indexed by time.

Definition A.28 *Let* $(\Omega, \mathcal{A}, \Pr)$ *be a probability space and let* T *be a totally ordered set (e.g. time). A stochastic process is a function* $X_t(\omega)$ *from* $T \times (\Omega, \mathcal{A}, \Pr)$ *to the set of real numbers such that* $\{X_t \in B\}$ *is an event for any Borel set* B *at each time* t.

For each fixed ω, the function $X_t(\omega)$ is called a sample path of the stochastic process X_t. A stochastic process X_t is said to be sample-continuous if almost all sample paths are continuous with respect to t.

Definition A.29 *A stochastic process* X_t *is said to have independent increments if*

$$X_{t_0},\ X_{t_1} - X_{t_0},\ X_{t_2} - X_{t_1},\ \cdots,\ X_{t_k} - X_{t_{k-1}} \tag{A.111}$$

are independent random variables where t_0 *is the initial time and* $t_1, t_2, \cdots, t_k$ *are any times with* $t_0 < t_1 < \cdots < t_k$.

Definition A.30 *A stochastic process* X_t *is said to have stationary increments if, for any given* $t > 0$*, the increments* $X_{s+t} - X_s$ *are identically distributed random variables for all* $s > 0$.

A stationary independent increment process is a stochastic process that has not only independent increments but also stationary increments. If X_t is a stationary independent increment process, then

$$Y_t = aX_t + b$$

is also a stationary independent increment process for any numbers a and b.

Renewal Process

Let ξ_i denote the times between the $(i-1)$th and the ith events, known as the interarrival times, $i = 1, 2, \cdots$, respectively. Define $S_0 = 0$ and

$$S_n = \xi_1 + \xi_2 + \cdots + \xi_n, \quad \forall n \geq 1. \tag{A.112}$$

Then S_n can be regarded as the waiting time until the occurrence of the nth event after time $t = 0$.

Definition A.31 *Let* $\xi_1, \xi_2, \cdots$ *be iid positive interarrival times. Define* $S_0 = 0$ *and* $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ *for* $n \geq 1$. *Then the stochastic process*

$$N_t = \max_{n \geq 0} \{n \mid S_n \leq t\} \tag{A.113}$$

is called a renewal process.

A renewal process is called a *Poisson process* with rate β if the interarrival times are exponential random variables with a common probability density function,

$$\phi(x) = \frac{1}{\beta} \exp\left(-\frac{x}{\beta}\right), \quad x \geq 0. \tag{A.114}$$

Wiener Process

In 1827 Robert Brown observed irregular movement of pollen grain suspended in liquid. This movement is now known as Brownian motion. In 1923 Norbert Wiener modeled Brownian motion by the following Wiener process.

Definition A.32 *A stochastic process W_t is said to be a standard Wiener process if*
(i) $W_0 = 0$ and almost all sample paths are continuous,
(ii) W_t has stationary and independent increments,
(iii) every increment $W_{s+t} - W_s$ is a normal random variable with expected value 0 and variance t.

Note that the lengths of almost all sample paths of Wiener process are infinitely long during any fixed time interval, and are differentiable nowhere. Furthermore, the squared variation of Wiener process on $[0, t]$ is equal to t both in mean square and almost surely.

A.15 Stochastic Calculus

Ito calculus, named after Kiyoshi Ito, is the most popular topic of stochastic calculus. The central concept is the Ito integral that allows one to integrate a stochastic process with respect to Wiener process. This section provides a brief introduction to Ito calculus.

Definition A.33 *Let X_t be a stochastic process and let W_t be a standard Wiener process. For any partition of closed interval $[a, b]$ with $a = t_1 < t_2 < \cdots < t_{k+1} = b$, the mesh is written as*

$$\Delta = \max_{1 \le i \le k} |t_{i+1} - t_i|.$$

Then Ito integral of X_t with respect to W_t is

$$\int_a^b X_t \mathrm{d}W_t = \lim_{\Delta \to 0} \sum_{i=1}^{k} X_{t_i} \cdot (W_{t_{i+1}} - W_{t_i}) \tag{A.115}$$

provided that the limit exists in mean square and is a random variable.

Example A.28: Let W_t be a standard Wiener process. It follows from the definition of Ito integral that

$$\int_0^s \mathrm{d}W_t = W_s,$$

$$\int_0^s W_t \mathrm{d}W_t = \frac{1}{2}W_s^2 - \frac{1}{2}s.$$

Definition A.34 *Let W_t be a standard Wiener process and let Z_t be a stochastic process. If there exist two stochastic processes μ_t and σ_t such that*

$$Z_t = Z_0 + \int_0^t \mu_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s \tag{A.116}$$

for any $t \geq 0$, then Z_t is called an Ito process with drift μ_t and diffusion σ_t. Furthermore, Z_t has a stochastic differential

$$\mathrm{d}Z_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t. \tag{A.117}$$

Theorem A.43 *(Ito Formula) Let W_t be a standard Wiener process, and let $h(t, w)$ be a twice continuously differentiable function. Then $X_t = h(t, W_t)$ is an Ito process and has a stochastic differential*

$$\mathrm{d}X_t = \frac{\partial h}{\partial t}(t, W_t)\mathrm{d}t + \frac{\partial h}{\partial w}(t, W_t)\mathrm{d}W_t + \frac{1}{2}\frac{\partial^2 h}{\partial w^2}(t, W_t)\mathrm{d}t. \tag{A.118}$$

Proof: Since the function h is twice continuously differentiable, by using Taylor series expansion, the infinitesimal increment of X_t has a second-order approximation

$$\begin{aligned} \Delta X_t &= \frac{\partial h}{\partial t}(t, W_t)\Delta t + \frac{\partial h}{\partial b}(t, W_t)\Delta W_t + \frac{1}{2}\frac{\partial^2 h}{\partial b^2}(t, W_t)(\Delta W_t)^2 \\ &+ \frac{1}{2}\frac{\partial^2 h}{\partial t^2}(t, W_t)(\Delta t)^2 + \frac{\partial^2 h}{\partial t \partial b}(t, W_t)\Delta t \Delta W_t. \end{aligned}$$

Since we can ignore the terms $(\Delta t)^2$ and $\Delta t \Delta W_t$ and replace $(\Delta W_t)^2$ with Δt, the Ito formula is obtained because it makes

$$X_s = X_0 + \int_0^s \frac{\partial h}{\partial t}(t, W_t)\mathrm{d}t + \int_0^s \frac{\partial h}{\partial b}(t, W_t)\mathrm{d}W_t + \frac{1}{2}\int_0^s \frac{\partial^2 h}{\partial b^2}(t, W_t)\mathrm{d}t$$

for any $s \geq 0$.

Example A.29: Ito formula is the fundamental theorem of stochastic calculus. Applying Ito formula, we obtain

$$\mathrm{d}(tW_t) = W_t \mathrm{d}t + t\mathrm{d}W_t,$$

$$\mathrm{d}(W_t^2) = 2W_t \mathrm{d}W_t + \mathrm{d}t.$$

A.16 Stochastic Differential Equation

In 1940s Kiyoshi Ito invented a type of stochastic differential equation that is a differential equation driven by Wiener process. This section provides a brief introduction to stochastic differential equation.

Definition A.35 *Suppose W_t is a standard Wiener process, and f and g are two functions. Then*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}W_t \tag{A.119}$$

is called a stochastic differential equation. A solution is an Ito process X_t that satisfies (A.119) identically in t.

Example A.30: Let W_t be a standard Wiener process. Then the stochastic differential equation

$$\mathrm{d}X_t = a\mathrm{d}t + b\mathrm{d}W_t$$

has a solution

$$X_t = at + bW_t.$$

Example A.31: Let W_t be a standard Wiener process. Then the stochastic differential equation

$$\mathrm{d}X_t = aX_t\mathrm{d}t + bX_t\mathrm{d}W_t$$

has a solution

$$X_t = \exp\left(\left(a - \frac{b^2}{2}\right)t + bW_t\right).$$

Theorem A.44 *(Existence and Uniqueness Theorem) The stochastic differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}W_t \tag{A.120}$$

has a unique solution if the coefficients $f(t, x)$ and $g(t, x)$ satisfy linear growth condition

$$|f(t, x)| + |g(t, x)| \le L(1 + |x|), \quad \forall x \in \Re, t \ge 0 \tag{A.121}$$

and Lipschitz condition

$$|f(t, x) - f(t, y)| + |g(t, x) - g(t, y)| \le L|x - y|, \quad \forall x, y \in \Re, t \ge 0 \tag{A.122}$$

for some constant L. Moreover, the solution is sample-continuous.

Theorem A.45 *(Feynman-Kac Formula) Consider the stochastic differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}W_t. \tag{A.123}$$

For any measurable function $h(x)$ and fixed $T > 0$, the function

$$U(t, x) = E\left[\int_t^T h(X_s)\mathrm{d}s \;\Big|\; X_t = x\right] \tag{A.124}$$

is the solution of the partial differential equation

$$\frac{\partial U}{\partial t}(t, x) + f(t, x)\frac{\partial U}{\partial x}(t, x) + \frac{1}{2}g^2(t, x)\frac{\partial^2 U}{\partial x^2}(t, x) + h(x) = 0 \tag{A.125}$$

with the terminal condition

$$U(T, x) = 0. \tag{A.126}$$

Appendix B

Chance Theory

Uncertainty and randomness are two basic types of indeterminacy. Chance theory was pioneered by Liu [149] in 2013 for modeling complex systems with not only uncertainty but also randomness. This appendix will introduce the concepts of chance measure, uncertain random variable, chance distribution, operational law, expected value, variance, and law of large numbers. As applications of chance theory, this appendix will also provide uncertain random programming, uncertain random risk analysis, uncertain random reliability analysis, uncertain random graph, uncertain random network, and uncertain random process.

B.1 Chance Measure

Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and let $(\Omega, \mathcal{A}, \Pr)$ be a probability space. Then the product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ is called a *chance space*. Essentially, it is another triplet,

$$(\Gamma \times \Omega, \mathcal{L} \times \mathcal{A}, \mathcal{M} \times \Pr) \tag{B.1}$$

where $\Gamma \times \Omega$ is the universal set, $\mathcal{L} \times \mathcal{A}$ is the product σ-algebra, and $\mathcal{M} \times \Pr$ is the product measure.

The universal set $\Gamma \times \Omega$ is clearly the set of all ordered pairs of the form (γ, ω), where $\gamma \in \Gamma$ and $\omega \in \Omega$. That is,

$$\Gamma \times \Omega = \{(\gamma, \omega) \mid \gamma \in \Gamma, \omega \in \Omega\}. \tag{B.2}$$

The product σ-algebra $\mathcal{L} \times \mathcal{A}$ is the smallest σ-algebra containing measurable rectangles of the form $\Lambda \times A$, where $\Lambda \in \mathcal{L}$ and $A \in \mathcal{A}$. Any element in $\mathcal{L} \times \mathcal{A}$ is called an *event* in the chance space.

What is the product measure $\mathcal{M} \times \Pr$? In order to answer this question, let us consider an event Θ in $\mathcal{L} \times \mathcal{A}$. For each $\omega \in \Omega$, the set

$$\Theta_\omega = \{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \tag{B.3}$$

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5

is clearly an event in $\mathcal{L}$. Thus the uncertain measure $\mathcal{M}\{\Theta_\omega\}$ exists for each $\omega \in \Omega$. However, unfortunately, $\mathcal{M}\{\Theta_\omega\}$ is not necessarily a measurable function with respect to ω. In other words, for a real number x, the set

$$\Theta_x^* = \{\omega \in \Omega \mid \mathcal{M}\{\Theta_\omega\} \geq x\} \tag{B.4}$$

is a subset of Ω but not necessarily an event in $\mathcal{A}$. Thus the probability measure $\Pr\{\Theta_x^*\}$ does not necessarily exist. In this case, we assign

$$\Pr\{\Theta_x^*\} = \begin{cases} \inf\limits_{A\in\mathcal{A}, A\supset\Theta_x^*} \Pr\{A\}, & \text{if } \inf\limits_{A\in\mathcal{A}, A\supset\Theta_x^*} \Pr\{A\} < 0.5 \\ \sup\limits_{A\in\mathcal{A}, A\subset\Theta_x^*} \Pr\{A\}, & \text{if } \sup\limits_{A\in\mathcal{A}, A\subset\Theta_x^*} \Pr\{A\} > 0.5 \\ 0.5, & \text{otherwise} \end{cases} \tag{B.5}$$

in the light of maximum uncertainty principle. This ensures the probability measure $\Pr\{\Theta_x^*\}$ exists for any real number x. Now it is ready to define $\mathcal{M} \times \Pr$ of Θ as the expected value of $\mathcal{M}\{\Theta_\omega\}$ with respect to $\omega \in \Omega$, i.e.,

$$\int_0^1 \Pr\{\Theta_x^*\}\mathrm{d}x. \tag{B.6}$$

Note that the above-mentioned integral is neither an uncertain measure nor a probability measure. We will call it chance measure and represent it by $\mathrm{Ch}\{\Theta\}$.

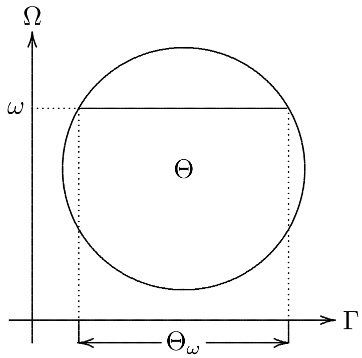


Figure B.1: An Event Θ in $\mathcal{L} \times \mathcal{A}$

Definition B.1 *(Liu [149]) Let* $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$ *be a chance space, and let* $\Theta \in \mathcal{L} \times \mathcal{A}$ *be an event. Then the chance measure of* Θ *is defined as*

$$\mathrm{Ch}\{\Theta\} = \int_0^1 \Pr\Big\{ \underbrace{\omega \in \Omega \mid \mathcal{M}\{\overbrace{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta}^{\Theta_\omega}\} \geq x}_{\Theta_x^*} \Big\}\mathrm{d}x. \tag{B.7}$$

Theorem B.1 *(Liu [149]) Let* $(\Gamma,\mathcal{L},\mathcal{M})\times(\Omega,\mathcal{A},\Pr)$ *be a chance space. Then*

$$\text{Ch}\{\Lambda\times A\}=\mathcal{M}\{\Lambda\}\times\Pr\{A\} \tag{B.8}$$

for any $\Lambda\in\mathcal{L}$ *and any* $A\in\mathcal{A}$*. Especially, we have*

$$\text{Ch}\{\emptyset\}=0,\quad \text{Ch}\{\Gamma\times\Omega\}=1. \tag{B.9}$$

Proof: Let us first prove the identity (B.8). For each $\omega\in\Omega$, we immediately have

$$\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Lambda\times A\}=\Lambda$$

and

$$\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Lambda\times A\}=\mathcal{M}\{\Lambda\}.$$

For any real number x, if $\mathcal{M}\{\Lambda\}\ge x$, then

$$\Pr\left\{\omega\in\Omega\,|\,\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Lambda\times A\}\ge x\right\}=\Pr\{A\}.$$

If $\mathcal{M}\{\Lambda\}<x$, then

$$\Pr\left\{\omega\in\Omega\,|\,\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Lambda\times A\}\ge x\right\}=\Pr\{\emptyset\}=0.$$

Thus

$$\begin{aligned}\text{Ch}\{\Lambda\times A\}&=\int_0^1\Pr\left\{\omega\in\Omega\,|\,\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Lambda\times A\}\ge x\right\}\mathrm{d}x\\&=\int_0^{\mathcal{M}\{\Lambda\}}\Pr\{A\}\mathrm{d}x+\int_{\mathcal{M}\{\Lambda\}}^1 0\mathrm{d}x=\mathcal{M}\{\Lambda\}\times\Pr\{A\}.\end{aligned}$$

Furthermore, it follows from (B.8) that

$$\text{Ch}\{\emptyset\}=\mathcal{M}\{\emptyset\}\times\Pr\{\emptyset\}=0,$$

$$\text{Ch}\{\Gamma\times\Omega\}=\mathcal{M}\{\Gamma\}\times\Pr\{\Omega\}=1.$$

The theorem is thus verified.

Theorem B.2 *(Liu [149], Monotonicity Theorem) Let* $(\Gamma,\mathcal{L},\mathcal{M})\times(\Omega,\mathcal{A},\Pr)$ *be a chance space. Then the chance measure* $\text{Ch}\{\Theta\}$ *is a monotone increasing function with respect to* Θ*.*

Proof: Let Θ_1 and Θ_2 be two events with $\Theta_1\subset\Theta_2$. Then for each ω, we have

$$\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta_1\}\subset\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta_2\}$$

and

$$\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta_1\}\le\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta_2\}.$$

Thus for any real number x, we have

$$\begin{aligned}&\Pr\{\omega\in\Omega\,|\,\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta_1\}\ge x\}\\&\quad\le\Pr\{\omega\in\Omega\,|\,\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta_2\}\ge x\}.\end{aligned}$$

By the definition of chance measure, we get

$$\begin{aligned}\text{Ch}\{\Theta_1\}&=\int_0^1\Pr\{\omega\in\Omega\,|\,\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta_1\}\ge x\}\,\mathrm{d}x\\&\le\int_0^1\Pr\{\omega\in\Omega\,|\,\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta_2\}\ge x\}\,\mathrm{d}x=\text{Ch}\{\Theta_2\}.\end{aligned}$$

That is, $\text{Ch}\{\Theta\}$ is a monotone increasing function with respect to Θ. The theorem is thus verified.

Theorem B.3 *(Liu [149], Duality Theorem) The chance measure is self-dual. That is, for any event Θ, we have*

$$\text{Ch}\{\Theta\}+\text{Ch}\{\Theta^c\}=1. \tag{B.10}$$

Proof: Since both uncertain measure and probability measure are self-dual, we have

$$\begin{aligned}\text{Ch}\{\Theta\}&=\int_0^1\Pr\{\omega\in\Omega\,|\,\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta\}\ge x\}\,\mathrm{d}x\\&=\int_0^1\Pr\{\omega\in\Omega\,|\,\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta^c\}\le 1-x\}\,\mathrm{d}x\\&=\int_0^1(1-\Pr\{\omega\in\Omega\,|\,\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta^c\}>1-x\})\,\mathrm{d}x\\&=1-\int_0^1\Pr\{\omega\in\Omega\,|\,\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta^c\}>x\}\,\mathrm{d}x\\&=1-\text{Ch}\{\Theta^c\}.\end{aligned}$$

That is, $\text{Ch}\{\Theta\}+\text{Ch}\{\Theta^c\}=1$, i.e., the chance measure is self-dual.

Theorem B.4 *(Hou [63], Subadditivity Theorem) The chance measure is subadditive. That is, for any countable sequence of events $\Theta_1,\Theta_2,\cdots$, we have*

$$\text{Ch}\left\{\bigcup_{i=1}^{\infty}\Theta_i\right\}\le\sum_{i=1}^{\infty}\text{Ch}\{\Theta_i\}. \tag{B.11}$$

Proof: For each ω, it follows from the subadditivity of uncertain measure that

$$\mathcal{M}\left\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\bigcup_{i=1}^{\infty}\Theta_i\right\}\le\sum_{i=1}^{\infty}\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta_i\}.$$

Thus for any real number x, we have

$$\begin{aligned}&\Pr\left\{\omega\in\Omega\,|\,\mathcal{M}\left\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\bigcup_{i=1}^{\infty}\Theta_i\right\}\ge x\right\}\\&\le\Pr\left\{\omega\in\Omega\,|\,\sum_{i=1}^{\infty}\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta_i\}\ge x\right\}.\end{aligned}$$

By the definition of chance measure, we get

$$\begin{aligned}\mathrm{Ch}\left\{\bigcup_{i=1}^{\infty}\Theta_i\right\}&=\int_0^1\Pr\left\{\omega\in\Omega\,|\,\mathcal{M}\left\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\bigcup_{i=1}^{\infty}\Theta_i\right\}\ge x\right\}\mathrm{d}x\\&\le\int_0^1\Pr\left\{\omega\in\Omega\,|\,\sum_{i=1}^{\infty}\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta_i\}\ge x\right\}\mathrm{d}x\\&\le\int_0^{+\infty}\Pr\left\{\omega\in\Omega\,|\,\sum_{i=1}^{\infty}\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta_i\}\ge x\right\}\mathrm{d}x\\&=\sum_{i=1}^{\infty}\int_0^1\Pr\left\{\omega\in\Omega\,|\,\mathcal{M}\{\gamma\in\Gamma\,|\,(\gamma,\omega)\in\Theta_i\}\ge x\right\}\mathrm{d}x\\&=\sum_{i=1}^{\infty}\mathrm{Ch}\{\Theta_i\}.\end{aligned}$$

That is, the chance measure is subadditive.

B.2 Uncertain Random Variable

Theoretically, an uncertain random variable is a measurable function on the chance space. It is usually used to deal with measurable functions of uncertain variables and random variables.

Definition B.2 *(Liu [149]) An uncertain random variable is a function* ξ *from a chance space* $(\Gamma,\mathcal{L},\mathcal{M})\times(\Omega,\mathcal{A},\Pr)$ *to the set of real numbers such that* $\{\xi\in B\}$ *is an event in* $\mathcal{L}\times\mathcal{A}$ *for any Borel set* B*.*

Remark B.1: An uncertain random variable $\xi(\gamma,\omega)$ degenerates to a random variable if it does not vary with γ. Thus a random variable is a special uncertain random variable.

Remark B.2: An uncertain random variable $\xi(\gamma,\omega)$ degenerates to an uncertain variable if it does not vary with ω. Thus an uncertain variable is a special uncertain random variable.

Theorem B.5 *Let $\xi_1, \xi_2, \cdots, \xi_n$ be uncertain random variables on the chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$, and let $f : \Re^n \to \Re$ be a measurable function. Then $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is an uncertain random variable determined by*

$$\xi(\gamma, \omega) = f(\xi_1(\gamma, \omega), \xi_2(\gamma, \omega), \cdots, \xi_n(\gamma, \omega)) \tag{B.12}$$

for all $(\gamma, \omega) \in \Gamma \times \Omega$.

Proof: Since $\xi_1, \xi_2, \cdots, \xi_n$ are uncertain random variables, we know that they are measurable functions on the chance space, and $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ is also a measurable function. Hence ξ is an uncertain random variable.

Example B.1: A random variable η plus an uncertain variable τ makes an uncertain random variable ξ, i.e.,

$$\xi(\gamma, \omega) = \eta(\omega) + \tau(\gamma) \tag{B.13}$$

for all $(\gamma, \omega) \in \Gamma \times \Omega$.

Example B.2: Let $\eta_1, \eta_2, \cdots, \eta_m$ be random variables, and let $\tau_1, \tau_2, \cdots, \tau_n$ be uncertain variables. If f is a measurable function, then

$$\xi = f(\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n) \tag{B.14}$$

is an uncertain random variable determined by

$$\xi(\gamma, \omega) = f(\eta_1(\omega), \eta_2(\omega), \cdots, \eta_m(\omega), \tau_1(\gamma), \tau_2(\gamma), \cdots, \tau_n(\gamma)) \tag{B.15}$$

for all $(\gamma, \omega) \in \Gamma \times \Omega$.

Theorem B.6 *(Liu [149]) Let ξ be an uncertain random variable on the chance space $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \Pr)$, and let B be a Borel set. Then $\{\xi \in B\}$ is an uncertain random event with chance measure*

$$\text{Ch}\{\xi \in B\} = \int_0^1 \Pr\left\{\omega \in \Omega \,|\, \mathcal{M}\{\gamma \in \Gamma \,|\, \xi(\gamma, \omega) \in B\} \geq x\right\} \mathrm{d}x. \tag{B.16}$$

Proof: Since $\{\xi \in B\}$ is an event in the chance space, the equation (B.16) follows from Definition B.1 immediately.

Remark B.3: If the uncertain random variable degenerates to a random variable η, then $\text{Ch}\{\eta \in B\} = \text{Ch}\{\Gamma \times (\eta \in B)\} = \mathcal{M}\{\Gamma\} \times \Pr\{\eta \in B\} = \Pr\{\eta \in B\}$. That is,

$$\text{Ch}\{\eta \in B\} = \Pr\{\eta \in B\}. \tag{B.17}$$

If the uncertain random variable degenerates to an uncertain variable τ, then $\text{Ch}\{\tau \in B\} = \text{Ch}\{(\tau \in B) \times \Omega\} = \mathcal{M}\{\tau \in B\} \times \Pr\{\Omega\} = \mathcal{M}\{\tau \in B\}$. That is,

$$\text{Ch}\{\tau \in B\} = \mathcal{M}\{\tau \in B\}. \tag{B.18}$$

Theorem B.7 *(Liu [149]) Let ξ be an uncertain random variable. Then the chance measure* $\text{Ch}\{\xi \in B\}$ *is a monotone increasing function of B and*

$$\text{Ch}\{\xi \in \emptyset\} = 0, \quad \text{Ch}\{\xi \in \Re\} = 1. \tag{B.19}$$

Proof: Let B_1 and B_2 be Borel sets with $B_1 \subset B_2$. Then we immediately have $\{\xi \in B_1\} \subset \{\xi \in B_2\}$. It follows from the monotonicity of chance measure that

$$\text{Ch}\{\xi \in B_1\} \le \text{Ch}\{\xi \in B_2\}.$$

Hence $\text{Ch}\{\xi \in B\}$ is a monotone increasing function of B. Furthermore, we have

$$\text{Ch}\{\xi \in \emptyset\} = \text{Ch}\{\emptyset\} = 0,$$

$$\text{Ch}\{\xi \in \Re\} = \text{Ch}\{\Gamma \times \Omega\} = 1.$$

The theorem is verified.

Theorem B.8 *(Liu [149]) Let ξ be an uncertain random variable. Then for any Borel set B, we have*

$$\text{Ch}\{\xi \in B\} + \text{Ch}\{\xi \in B^c\} = 1. \tag{B.20}$$

Proof: It follows from $\{\xi \in B\}^c = \{\xi \in B^c\}$ and the duality of chance measure immediately.

B.3 Chance Distribution

Definition B.3 *(Liu [149]) Let ξ be an uncertain random variable. Then its chance distribution is defined by*

$$\Phi(x) = \text{Ch}\{\xi \le x\} \tag{B.21}$$

for any $x \in \Re$.

Example B.3: As a special uncertain random variable, the chance distribution of a random variable η is just its probability distribution, that is,

$$\Phi(x) = \text{Ch}\{\eta \le x\} = \Pr\{\eta \le x\}. \tag{B.22}$$

Example B.4: As a special uncertain random variable, the chance distribution of an uncertain variable τ is just its uncertainty distribution, that is,

$$\Phi(x) = \text{Ch}\{\tau \le x\} = \mathcal{M}\{\tau \le x\}. \tag{B.23}$$

Theorem B.9 *(Liu [149], Sufficient and Necessary Condition for Chance Distribution) A function $\Phi : \Re \to [0,1]$ is a chance distribution if and only if it is a monotone increasing function except $\Phi(x) \equiv 0$ and $\Phi(x) \equiv 1$.*

Proof: Assume Φ is a chance distribution of uncertain random variable ξ. Let x_1 and x_2 be two real numbers with $x_1 < x_2$. It follows from Theorem B.7 that

$$\Phi(x_1) = \text{Ch}\{\xi \le x_1\} \le \text{Ch}\{\xi \le x_2\} = \Phi(x_2).$$

Hence the chance distribution Φ is a monotone increasing function. Furthermore, if $\Phi(x) \equiv 0$, then

$$\int_0^1 \Pr\{\omega \in \Omega \,|\, \mathcal{M}\{\gamma \in \Gamma \,|\, \xi(\gamma, \omega) \le x\} \ge r\} \,\text{d}r \equiv 0.$$

Thus for almost all $\omega \in \Omega$, we have

$$\mathcal{M}\{\gamma \in \Gamma \,|\, \xi(\gamma, \omega) \le x\} \equiv 0, \quad \forall x \in \Re$$

which is in contradiction to the asymptotic theorem, and then $\Phi(x) \not\equiv 0$ is verified. Similarly, if $\Phi(x) \equiv 1$, then

$$\int_0^1 \Pr\{\omega \in \Omega \,|\, \mathcal{M}\{\gamma \in \Gamma \,|\, \xi(\gamma, \omega) \le x\} \ge r\} \,\text{d}r \equiv 1.$$

Thus for almost all $\omega \in \Omega$, we have

$$\mathcal{M}\{\gamma \in \Gamma \,|\, \xi(\gamma, \omega) \le x\} \equiv 1, \quad \forall x \in \Re$$

which is also in contradiction to the asymptotic theorem, and then $\Phi(x) \not\equiv 1$ is proved.

Conversely, suppose $\Phi : \Re \to [0, 1]$ is a monotone increasing function but $\Phi(x) \not\equiv 0$ and $\Phi(x) \not\equiv 1$. It follows from Peng-Iwamura theorem that there is an uncertain variable whose uncertainty distribution is just $\Phi(x)$. Since an uncertain variable is a special uncertain random variable, we know that Φ is a chance distribution.

Theorem B.10 *(Liu [149], Chance Inversion Theorem) Let ξ be an uncertain random variable with chance distribution Φ. Then for any real number x, we have*

$$\text{Ch}\{\xi \le x\} = \Phi(x), \quad \text{Ch}\{\xi > x\} = 1 - \Phi(x). \tag{B.24}$$

Proof: The equation $\text{Ch}\{\xi \le x\} = \Phi(x)$ follows from the definition of chance distribution immediately. By using the duality of chance measure, we get

$$\text{Ch}\{\xi > x\} = 1 - \text{Ch}\{\xi \le x\} = 1 - \Phi(x).$$

Remark B.4: When the chance distribution Φ is a continuous function, we also have

$$\text{Ch}\{\xi < x\} = \Phi(x), \quad \text{Ch}\{\xi \ge x\} = 1 - \Phi(x). \tag{B.25}$$

B.4 Operational Law

Assume $\eta_1, \eta_2, \cdots, \eta_m$ are independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and $\tau_1, \tau_2, \cdots, \tau_n$ are independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. What is the chance distribution of the uncertain random variable

$$\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)? \tag{B.26}$$

This section will provide an operational law to answer this question.

Theorem B.11 *(Liu [150]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively, and let $\tau_1, \tau_2, \cdots, \tau_n$ be uncertain variables (not necessarily independent). Then the uncertain random variable*

$$\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \tag{B.27}$$

has a chance distribution

$$\Phi(x) = \int_{\Re^m} \mathcal{M}\{f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n) \le x\} \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \tag{B.28}$$

for any number x.

Proof: It follows from Theorem B.6 that the uncertain random variable ξ has a chance distribution

$$\begin{aligned} \Phi(x) &= \int_0^1 \Pr\{\omega \in \Omega \,|\, \mathcal{M}\{\gamma \in \Gamma \,|\, \xi(\gamma, \omega) \le x\} \ge r\} \,\mathrm{d}r \\ &= \int_0^1 \Pr\{\omega \in \Omega \,|\, \mathcal{M}\{f(\eta_1(\omega), \cdots, \eta_m(\omega), \tau_1, \cdots, \tau_n) \le x\} \ge r\} \,\mathrm{d}r \\ &= \int_{\Re^m} \mathcal{M}\{f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n) \le x\} \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m). \end{aligned}$$

The theorem is verified.

Theorem B.12 *(Liu [150]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively, and let $\tau_1, \tau_2, \cdots, \tau_n$ be uncertain variables (not necessarily independent). Then the uncertain random variable*

$$\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \tag{B.29}$$

has a chance distribution

$$\Phi(x) = \int_{\Re^m} F(x; y_1, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \tag{B.30}$$

where $F(x; y_1, \cdots, y_m)$ is the uncertainty distribution of the uncertain variable $f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)$ for any real numbers $y_1, \cdots, y_m$.

Proof: For any given numbers $y_1, \cdots, y_m$, it follows from the operational law of uncertain variables that $f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)$ is an uncertain variable with uncertainty distribution $F(x; y_1, \cdots, y_m)$. By using (B.28), the chance distribution of ξ is

$$\begin{aligned}\Phi(x) &= \int_{\Re^m} \mathcal{M}\{f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n) \le x\} \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \\ &= \int_{\Re^m} F(x; y_1, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m)\end{aligned}$$

that is just (B.30). The theorem is verified.

Remark B.5: Let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If the function

$$f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$$

is strictly increasing with respect to $\tau_1, \cdots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \cdots, \tau_n$, then $F^{-1}(\alpha; y_1, \cdots, y_m)$ is equal to

$$f(y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha))$$

from which we may derive the uncertainty distribution $F(x; y_1, \cdots, y_m)$.

Exercise B.1: Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. Show that the sum

$$\xi = \eta_1 + \eta_2 + \cdots + \eta_m + \tau_1 + \tau_2 + \cdots + \tau_n \tag{B.31}$$

has a chance distribution

$$\Phi(x) = \int_{-\infty}^{+\infty} \Upsilon(x - y) \mathrm{d}\Psi(y) \tag{B.32}$$

where

$$\Psi(y) = \int_{y_1 + y_2 + \cdots + y_m \le y} \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \cdots \mathrm{d}\Psi_m(y_m) \tag{B.33}$$

is the probability distribution of $\eta_1 + \eta_2 + \cdots + \eta_m$, and

$$\Upsilon(z) = \sup_{z_1 + z_2 + \cdots + z_n = z} \Upsilon_1(z_1) \wedge \Upsilon_2(z_2) \wedge \cdots \wedge \Upsilon_n(z_n) \tag{B.34}$$

is the uncertainty distribution of $\tau_1 + \tau_2 + \cdots + \tau_n$.

Exercise B.2: Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent positive random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$

be independent positive uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. Show that the product

$$\xi = \eta_1 \eta_2 \cdots \eta_m \tau_1 \tau_2 \cdots \tau_n \tag{B.35}$$

has a chance distribution

$$\Phi(x) = \int_0^{+\infty} \Upsilon(x/y) \mathrm{d}\Psi(y) \tag{B.36}$$

where

$$\Psi(y) = \int_{y_1 y_2 \cdots y_m \le y} \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \cdots \mathrm{d}\Psi_m(y_m) \tag{B.37}$$

is the probability distribution of $\eta_1 \eta_2 \cdots \eta_m$, and

$$\Upsilon(z) = \sup_{z_1 z_2 \cdots z_n = z} \Upsilon_1(z_1) \wedge \Upsilon_2(z_2) \wedge \cdots \wedge \Upsilon_n(z_n) \tag{B.38}$$

is the uncertainty distribution of $\tau_1 \tau_2 \cdots \tau_n$.

Exercise B.3: Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. Show that the minimum

$$\xi = \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_m \wedge \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n \tag{B.39}$$

has a chance distribution

$$\Phi(x) = \Psi(x) + \Upsilon(x) - \Psi(x)\Upsilon(x) \tag{B.40}$$

where

$$\Psi(x) = 1 - (1 - \Psi_1(x))(1 - \Psi_2(x)) \cdots (1 - \Psi_m(x)) \tag{B.41}$$

is the probability distribution of $\eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_m$, and

$$\Upsilon(x) = \Upsilon_1(x) \vee \Upsilon_2(x) \vee \cdots \vee \Upsilon_n(x) \tag{B.42}$$

is the uncertainty distribution of $\tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n$.

Exercise B.4: Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. Show that the maximum

$$\xi = \eta_1 \vee \eta_2 \vee \cdots \vee \eta_m \vee \tau_1 \vee \tau_2 \vee \cdots \vee \tau_n \tag{B.43}$$

has a chance distribution

$$\Phi(x) = \Psi(x)\Upsilon(x) \tag{B.44}$$

where

$$\Psi(x) = \Psi_1(x)\Psi_2(x)\cdots\Psi_m(x) \tag{B.45}$$

is the probability distribution of $\eta_1 \vee \eta_2 \vee \cdots \vee \eta_m$, and

$$\Upsilon(x) = \Upsilon_1(x) \wedge \Upsilon_2(x) \wedge \cdots \wedge \Upsilon_n(x) \tag{B.46}$$

is the uncertainty distribution of $\tau_1 \vee \tau_2 \vee \cdots \vee \tau_n$.

Some Useful Theorems

In many cases, it is required to calculate $\text{Ch}\{f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \le 0\}$. We may produce the chance distribution $\Phi(x)$ of $f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ by the operational law, and then the chance measure is just $\Phi(0)$. However, for convenience, we may use the following theorems.

Theorem B.13 *(Liu [151]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$ and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with regular uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If $f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ is strictly increasing with respect to $\tau_1, \cdots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \cdots, \tau_n$, then*

$$\text{Ch}\{f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \le 0\} = \int_{\Re^m} G(y_1, \cdots, y_m)\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_m(y_m)$$

where $G(y_1, \cdots, y_m)$ is the root α of the equation

$$f(y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)) = 0.$$

Proof: It follows from the definition of chance measure that for any numbers $y_1, \cdots, y_m$, the theorem is true if the function G is

$$G(y_1, \cdots, y_m) = \mathcal{M}\{f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n) \le 0\}.$$

Furthermore, by using Theorem 2.20, we know that G is just the root α. The theorem is proved.

Remark B.6: Sometimes, the equation may not have a root. In this case, if

$$f(y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)) < 0$$

for all α, then we set the root $\alpha = 1$; and if

$$f(y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)) > 0$$

for all α, then we set the root $\alpha = 0$.

Remark B.7: The root α may be estimated by the bisection method because $f(y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha))$ is a strictly increasing function with respect to α. See Figure B.2.

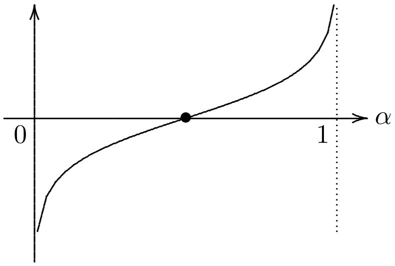


Figure B.2: $f(y_1,\cdots,y_m,\Upsilon_1^{-1}(\alpha),\cdots,\Upsilon_k^{-1}(\alpha),\Upsilon_{k+1}^{-1}(1-\alpha),\cdots,\Upsilon_n^{-1}(1-\alpha))$

Theorem B.14 *(Liu [151]) Let $\eta_1,\eta_2,\cdots,\eta_m$ be independent random variables with probability distributions $\Psi_1,\Psi_2,\cdots,\Psi_m$ and let $\tau_1,\tau_2,\cdots,\tau_n$ be independent uncertain variables with regular uncertainty distributions $\Upsilon_1,\Upsilon_2,\cdots,\Upsilon_n$, respectively. If $f(\eta_1,\cdots,\eta_m,\tau_1,\cdots,\tau_n)$ is strictly increasing with respect to $\tau_1,\cdots,\tau_k$ and strictly decreasing with respect to $\tau_{k+1},\cdots,\tau_n$, then*

$$\text{Ch}\{f(\eta_1,\cdots,\eta_m,\tau_1,\cdots,\tau_n)>0\}=\int_{\Re^m}G(y_1,\cdots,y_m)\mathrm{d}\Psi_1(y_1)\cdots\mathrm{d}\Psi_m(y_m)$$

where $G(y_1,\cdots,y_m)$ is the root α of the equation

$$f(y_1,\cdots,y_m,\Upsilon_1^{-1}(1-\alpha),\cdots,\Upsilon_k^{-1}(1-\alpha),\Upsilon_{k+1}^{-1}(\alpha),\cdots,\Upsilon_n^{-1}(\alpha))=0.$$

Proof: It follows from the definition of chance measure that for any numbers $y_1,\cdots,y_m$, the theorem is true if the function G is

$$G(y_1,\cdots,y_m)=\mathcal{M}\{f(y_1,\cdots,y_m,\tau_1,\cdots,\tau_n)>0\}.$$

Furthermore, by using Theorem 2.21, we know that G is just the root α. The theorem is proved.

Remark B.8: Sometimes, the equation may not have a root. In this case, if

$$f(y_1,\cdots,y_m,\Upsilon_1^{-1}(1-\alpha),\cdots,\Upsilon_k^{-1}(1-\alpha),\Upsilon_{k+1}^{-1}(\alpha),\cdots,\Upsilon_n^{-1}(\alpha))<0$$

for all α, then we set the root $\alpha=0$; and if

$$f(y_1,\cdots,y_m,\Upsilon_1^{-1}(1-\alpha),\cdots,\Upsilon_k^{-1}(1-\alpha),\Upsilon_{k+1}^{-1}(\alpha),\cdots,\Upsilon_n^{-1}(\alpha))>0$$

for all α, then we set the root $\alpha=1$.

Remark B.9: The root α may be estimated by the bisection method because $f(y_1,\cdots,y_m,\Upsilon_1^{-1}(1-\alpha),\cdots,\Upsilon_k^{-1}(1-\alpha),\Upsilon_{k+1}^{-1}(\alpha),\cdots,\Upsilon_n^{-1}(\alpha))$ is a strictly decreasing function with respect to α. See Figure B.3.

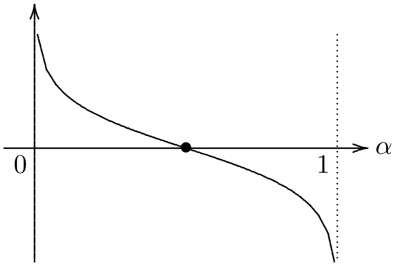


Figure B.3: $f(y_1,\cdots,y_m,\Upsilon_1^{-1}(1-\alpha),\cdots,\Upsilon_k^{-1}(1-\alpha),\Upsilon_{k+1}^{-1}(\alpha),\cdots,\Upsilon_n^{-1}(\alpha))$

Operational Law for Boolean System

Theorem B.15 *(Liu [150]) Assume $\eta_1,\eta_2,\cdots,\eta_m$ are independent Boolean random variables, i.e.,*

$$\eta_i = \begin{cases} \text{1 with probability measure } a_i \\ \text{0 with probability measure } 1-a_i \end{cases} \tag{B.47}$$

for $i=1,2,\cdots,m$, and $\tau_1,\tau_2,\cdots,\tau_n$ are independent Boolean uncertain variables, i.e.,

$$\tau_j = \begin{cases} \text{1 with uncertain measure } b_j \\ \text{0 with uncertain measure } 1-b_j \end{cases} \tag{B.48}$$

for $j=1,2,\cdots,n$. If f is a Boolean function (not necessarily monotone), then

$$\xi = f(\eta_1,\cdots,\eta_m,\tau_1,\cdots,\tau_n) \tag{B.49}$$

is a Boolean uncertain random variable such that

$$\text{Ch}\{\xi=1\} = \sum_{(x_1,\cdots,x_m)\in\{0,1\}^m} \left(\prod_{i=1}^m \mu_i(x_i)\right) f^*(x_1,\cdots,x_m) \tag{B.50}$$

where

$$f^*(x_1,\cdots,x_m) = \begin{cases} \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j), \\ \quad \textit{if } \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j) < 0.5 \\ 1 - \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=0} \min\limits_{1\le j\le n} \nu_j(y_j), \\ \quad \textit{if } \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j) \ge 0.5, \end{cases} \tag{B.51}$$

$$\mu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \qquad (i = 1, 2, \cdots, m), \tag{B.52}$$

$$\nu_j(y_j) = \begin{cases} b_j, & \text{if } y_j = 1 \\ 1 - b_j, & \text{if } y_j = 0 \end{cases} \qquad (j = 1, 2, \cdots, n). \tag{B.53}$$

Proof: At first, when $(x_1, \cdots, x_m)$ is given, $f(x_1, \cdots, x_m, \tau_1, \cdots, \tau_n)$ is essentially a Boolean function of uncertain variables. It follows from the operational law of uncertain variables that

$$\mathcal{M}\{f(x_1, \cdots, x_m, \tau_1, \cdots, \tau_n) = 1\} = f^*(x_1, \cdots, x_m)$$

that is determined by (B.51). On the other hand, it follows from the operational law of uncertain random variables that

$$\text{Ch}\{\xi = 1\} = \sum_{(x_1, \cdots, x_m) \in \{0,1\}^m} \left(\prod_{i=1}^m \mu_i(x_i) \right) \mathcal{M}\{f(x_1, \cdots, x_m, \tau_1, \cdots, \tau_n) = 1\}.$$

Thus (B.50) is verified.

Remark B.10: When the uncertain variables disappear, the operational law becomes

$$\Pr\{\xi = 1\} = \sum_{(x_1, x_2, \cdots, x_m) \in \{0,1\}^m} \left(\prod_{i=1}^m \mu_i(x_i) \right) f(x_1, x_2, \cdots, x_m). \tag{B.54}$$

Remark B.11: When the random variables disappear, the operational law becomes

$$\mathcal{M}\{\xi = 1\} = \begin{cases} \sup\limits_{f(y_1, y_2, \cdots, y_n) = 1} \min\limits_{1 \le j \le n} \nu_j(y_j), \\ \qquad \text{if} \sup\limits_{f(y_1, y_2, \cdots, y_n) = 1} \min\limits_{1 \le j \le n} \nu_j(y_j) < 0.5 \\ 1 - \sup\limits_{f(y_1, y_2, \cdots, y_n) = 0} \min\limits_{1 \le j \le n} \nu_j(y_j), \\ \qquad \text{if} \sup\limits_{f(y_1, y_2, \cdots, y_n) = 1} \min\limits_{1 \le j \le n} \nu_j(y_j) \ge 0.5. \end{cases} \tag{B.55}$$

Exercise B.5: Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent Boolean random variables defined by (B.47) and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent Boolean uncertain variables defined by (B.48). Then the minimum

$$\xi = \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_m \wedge \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n \tag{B.56}$$

is a Boolean uncertain random variable. Show that

$$\text{Ch}\{\xi = 1\} = a_1 a_2 \cdots a_m (b_1 \wedge b_2 \wedge \cdots \wedge b_n). \tag{B.57}$$

Exercise B.6: Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent Boolean random variables defined by (B.47) and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent Boolean uncertain variables defined by (B.48). Then the maximum

$$\xi = \eta_1 \vee \eta_2 \vee \cdots \vee \eta_m \vee \tau_1 \vee \tau_2 \vee \cdots \vee \tau_n \tag{B.58}$$

is a Boolean uncertain random variable. Show that

$$\text{Ch}\{\xi = 1\} = 1 - (1-a_1)(1-a_2)\cdots(1-a_m)(1 - b_1 \vee b_2 \vee \cdots \vee b_n). \tag{B.59}$$

Exercise B.7: Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent Boolean random variables defined by (B.47) and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent Boolean uncertain variables defined by (B.48). Then the kth largest value

$$\xi = k\text{-max}\,[\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n] \tag{B.60}$$

is a Boolean uncertain random variable. Show that

$$\text{Ch}\{\xi = 1\} = \sum_{(x_1,x_2,\cdots,x_m)\in\{0,1\}^m} \left(\prod_{i=1}^m \mu_i(x_i)\right) f^*(x_1, x_2, \cdots, x_m) \tag{B.61}$$

where

$$f^*(x_1, x_2, \cdots, x_m) = k\text{-max}\,[x_1, x_2, \cdots, x_m, b_1, b_2, \cdots, b_n], \tag{B.62}$$

$$\mu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \qquad (i = 1, 2, \cdots, m). \tag{B.63}$$

B.5 Expected Value

Definition B.4 *(Liu [149]) Let ξ be an uncertain random variable. Then its expected value is defined by*

$$E[\xi] = \int_0^{+\infty} \text{Ch}\{\xi \ge x\}\text{d}x - \int_{-\infty}^0 \text{Ch}\{\xi \le x\}\text{d}x \tag{B.64}$$

provided that at least one of the two integrals is finite.

Theorem B.16 *(Liu [149]) Let ξ be an uncertain random variable with chance distribution Φ. Then*

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x))\text{d}x - \int_{-\infty}^0 \Phi(x)\text{d}x. \tag{B.65}$$

Proof: It follows from the chance inversion theorem that for almost all numbers x, we have $\text{Ch}\{\xi \ge x\} = 1 - \Phi(x)$ and $\text{Ch}\{\xi \le x\} = \Phi(x)$. By using the definition of expected value operator, we obtain

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} \text{Ch}\{\xi \ge x\}\mathrm{d}x - \int_{-\infty}^0 \text{Ch}\{\xi \le x\}\mathrm{d}x \\ &= \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x. \end{aligned}$$

Thus we obtain the equation (B.65).

Theorem B.17 *Let ξ be an uncertain random variable with chance distribution Φ. Then*

$$E[\xi] = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x). \tag{B.66}$$

Proof: It follows from the change of variables of integral and Theorem B.16 that the expected value is

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x \\ &= \int_0^{+\infty} x\mathrm{d}\Phi(x) + \int_{-\infty}^0 x\mathrm{d}\Phi(x) = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x). \end{aligned}$$

The theorem is proved.

Theorem B.18 *Let ξ be an uncertain random variable with regular chance distribution Φ. Then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha. \tag{B.67}$$

Proof: It follows from the change of variables of integral and Theorem B.16 that the expected value is

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x \\ &= \int_{\Phi(0)}^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha + \int_0^{\Phi(0)} \Phi^{-1}(\alpha)\mathrm{d}\alpha = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha. \end{aligned}$$

The theorem is proved.

Theorem B.19 *(Liu [150]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, respectively, and let $\tau_1, \tau_2,$*

$\cdots, \tau_n$ *be uncertain variables (not necessarily independent), then the uncertain random variable*

$$\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \tag{B.68}$$

has an expected value

$$E[\xi] = \int_{\Re^m} E[f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)] \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \tag{B.69}$$

where $E[f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)]$ *is the expected value of the uncertain variable* $f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)$ *for any real numbers* $y_1, \cdots, y_m$.

Proof: For simplicity, we only prove the case $m = n = 2$. Write the uncertainty distribution of $f(y_1, y_2, \tau_1, \tau_2)$ by $F(x; y_1, y_2)$ for any real numbers y_1 and y_2. Then

$$E[f(y_1, y_2, \tau_1, \tau_2)] = \int_0^{+\infty} (1 - F(x; y_1, y_2)) \mathrm{d}x - \int_{-\infty}^0 F(x; y_1, y_2) \mathrm{d}x.$$

On the other hand, the uncertain random variable $\xi = f(\eta_1, \eta_2, \tau_1, \tau_2)$ has a chance distribution

$$\Phi(x) = \int_{\Re^2} F(x; y_1, y_2) \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2).$$

It follows from Theorem B.16 that

$$\begin{aligned}
E[\xi] &= \int_0^{+\infty} (1 - \Phi(x)) \mathrm{d}x - \int_{-\infty}^0 \Phi(x) \mathrm{d}x \\
&= \int_0^{+\infty} \left(1 - \int_{\Re^2} F(x; y_1, y_2) \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2)\right) \mathrm{d}x \\
&\quad - \int_{-\infty}^0 \int_{\Re^2} F(x; y_1, y_2) \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \mathrm{d}x \\
&= \int_{\Re^2} \left(\int_0^{+\infty} (1 - F(x; y_1, y_2)) \mathrm{d}x - \int_{-\infty}^0 F(x; y_1, y_2) \mathrm{d}x\right) \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \\
&= \int_{\Re^2} E[f(y_1, y_2, \tau_1, \tau_2)] \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2).
\end{aligned}$$

Thus the theorem is proved.

Example B.5: Let η be a random variable and let τ be an uncertain variable. Assume η has a probability distribution Ψ. It follows from Theorem B.19 that the uncertain random variable $\eta + \tau$ has an expected value

$$E[\eta + \tau] = \int_{\Re} E[y + \tau] \mathrm{d}\Psi(y) = \int_{\Re} (y + E[\tau]) \mathrm{d}\Psi(y) = E[\eta] + E[\tau].$$

That is,

$$E[\eta + \tau] = E[\eta] + E[\tau]. \tag{B.70}$$

Exercise B.8: Let η be a random variable and let τ be an uncertain variable. Assume η has a probability distribution Ψ. Show that

$$E[\eta\tau] = E[\eta]E[\tau]. \tag{B.71}$$

Theorem B.20 *(Liu [150]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If $f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ is a strictly increasing function or a strictly decreasing function with respect to $\tau_1, \cdots, \tau_n$, then the uncertain random variable*

$$\xi = f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \tag{B.72}$$

has an expected value

$$E[\xi] = \int_{\Re^m} \int_0^1 f(y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m).$$

Proof: Since $f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)$ is a strictly increasing function or a strictly decreasing function with respect to $\tau_1, \cdots, \tau_n$, we have

$$E[f(y_1, \cdots, y_m, \tau_1, \cdots, \tau_n)] = \int_0^1 f(y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) \mathrm{d}\alpha.$$

It follows from Theorem B.19 that the result holds.

Remark B.12: If $f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ is strictly increasing with respect to $\tau_1, \cdots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \cdots, \tau_n$, then the integrand in the formula of expected value $E[\xi]$ should be replaced with

$$f(y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha) \cdots, \Upsilon_n^{-1}(1-\alpha)).$$

Exercise B.9: Let η be a random variable with probability distribution Ψ, and let τ be an uncertain variable with uncertainty distribution Υ. Show that

$$E[\eta \vee \tau] = \int_{\Re} \int_0^1 \left(y \vee \Upsilon^{-1}(\alpha)\right) \mathrm{d}\alpha \mathrm{d}\Psi(y) \tag{B.73}$$

and

$$E[\eta \wedge \tau] = \int_{\Re} \int_0^1 \left(y \wedge \Upsilon^{-1}(\alpha)\right) \mathrm{d}\alpha \mathrm{d}\Psi(y). \tag{B.74}$$

Theorem B.21 *(Liu [150], Linearity of Expected Value Operator) Assume η_1 and η_2 are random variables (not necessarily independent), τ_1 and τ_2 are independent uncertain variables, and f_1 and f_2 are measurable functions. Then*

$$E[f_1(\eta_1, \tau_1) + f_2(\eta_2, \tau_2)] = E[f_1(\eta_1, \tau_1)] + E[f_2(\eta_2, \tau_2)]. \tag{B.75}$$

Proof: Since τ_1 and τ_2 are independent uncertain variables, for any real numbers y_1 and y_2, the functions $f_1(y_1,\tau_1)$ and $f_2(y_2,\tau_2)$ are also independent uncertain variables. Thus

$$E[f_1(y_1,\tau_1)+f_2(y_2,\tau_2)] = E[f_1(y_1,\tau_1)] + E[f_2(y_2,\tau_2)].$$

Let Ψ_1 and Ψ_2 be the probability distributions of random variables η_1 and η_2, respectively. Then we have

$$\begin{aligned}
&E[f_1(\eta_1,\tau_1)+f_2(\eta_2,\tau_2)]\\
&= \int_{\Re^2} E[f_1(y_1,\tau_1)+f_2(y_2,\tau_2)]\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)\\
&= \int_{\Re^2} (E[f_1(y_1,\tau_1)]+E[f_2(y_2,\tau_2)])\mathrm{d}\Psi_1(y_1)\mathrm{d}\Psi_2(y_2)\\
&= \int_{\Re} E[f_1(y_1,\tau_1)]\mathrm{d}\Psi_1(y_1) + \int_{\Re} E[f_2(y_2,\tau_2)]\mathrm{d}\Psi_2(y_2)\\
&= E[f_1(\eta_1,\tau_1)] + E[f_2(\eta_2,\tau_2)].
\end{aligned}$$

The theorem is proved.

Exercise B.10: Assume η_1 and η_2 are random variables, and τ_1 and τ_2 are independent uncertain variables. Show that

$$E[\eta_1\vee\tau_1+\eta_2\wedge\tau_2] = E[\eta_1\vee\tau_1]+E[\eta_2\wedge\tau_2]. \tag{B.76}$$

B.6 Variance

Definition B.5 *(Liu [149]) Let ξ be an uncertain random variable with finite expected value e. Then the variance of ξ is*

$$V[\xi] = E[(\xi-e)^2]. \tag{B.77}$$

Since $(\xi-e)^2$ is a nonnegative uncertain random variable, we also have

$$V[\xi] = \int_0^{+\infty} \mathrm{Ch}\{(\xi-e)^2\ge x\}\mathrm{d}x. \tag{B.78}$$

Theorem B.22 *(Liu [149]) If ξ is an uncertain random variable with finite expected value, a and b are real numbers, then*

$$V[a\xi+b] = a^2V[\xi]. \tag{B.79}$$

Proof: Let e be the expected value of ξ. Then $a\xi+b$ has an expected value $ae+b$. Thus the variance is

$$V[a\xi+b] = E[(a\xi+b-(ae+b))^2] = E[a^2(\xi-e)^2] = a^2V[\xi].$$

The theorem is verified.

Theorem B.23 *(Liu [149]) Let ξ be an uncertain random variable with expected value e. Then $V[\xi]=0$ if and only if $\mathrm{Ch}\{\xi=e\}=1$.*

Proof: We first assume $V[\xi]=0$. It follows from the equation (B.78) that

$$\int_0^{+\infty}\mathrm{Ch}\{(\xi-e)^2\ge x\}\mathrm{d}x=0$$

which implies $\mathrm{Ch}\{(\xi-e)^2\ge x\}=0$ for any $x>0$. Hence we have

$$\mathrm{Ch}\{(\xi-e)^2=0\}=1.$$

That is, $\mathrm{Ch}\{\xi=e\}=1$. Conversely, assume $\mathrm{Ch}\{\xi=e\}=1$. Then we immediately have $\mathrm{Ch}\{(\xi-e)^2=0\}=1$ and $\mathrm{Ch}\{(\xi-e)^2\ge x\}=0$ for any $x>0$. Thus

$$V[\xi]=\int_0^{+\infty}\mathrm{Ch}\{(\xi-e)^2\ge x\}\mathrm{d}x=0.$$

The theorem is proved.

How to Obtain Variance from Distributions?

Let ξ be an uncertain random variable with expected value e. If we only know its chance distribution Φ, then the variance

$$\begin{aligned}V[\xi]&=\int_0^{+\infty}\mathrm{Ch}\{(\xi-e)^2\ge x\}\mathrm{d}x\\&=\int_0^{+\infty}\mathrm{Ch}\{(\xi\ge e+\sqrt{x})\cup(\xi\le e-\sqrt{x})\}\mathrm{d}x\\&\le\int_0^{+\infty}(\mathrm{Ch}\{\xi\ge e+\sqrt{x}\}+\mathrm{Ch}\{\xi\le e-\sqrt{x}\})\mathrm{d}x\\&=\int_0^{+\infty}(1-\Phi(e+\sqrt{x})+\Phi(e-\sqrt{x}))\mathrm{d}x.\end{aligned}$$

Thus we have the following stipulation.

Stipulation B.1 *(Guo and Wang [57]) Let ξ be an uncertain random variable with chance distribution Φ and finite expected value e. Then*

$$V[\xi]=\int_0^{+\infty}(1-\Phi(e+\sqrt{x})+\Phi(e-\sqrt{x}))\mathrm{d}x. \tag{B.80}$$

Theorem B.24 *(Sheng and Yao [211]) Let ξ be an uncertain random variable with chance distribution Φ and finite expected value e. Then*

$$V[\xi]=\int_{-\infty}^{+\infty}(x-e)^2\mathrm{d}\Phi(x). \tag{B.81}$$

Proof: This theorem is based on Stipulation B.1 that says the variance of ξ is

$$V[\xi]=\int_0^{+\infty}(1-\Phi(e+\sqrt{y}))\mathrm{d}y+\int_0^{+\infty}\Phi(e-\sqrt{y})\mathrm{d}y.$$

Substituting $e+\sqrt{y}$ with x and y with $(x-e)^2$, the change of variables and integration by parts produce

$$\int_0^{+\infty}(1-\Phi(e+\sqrt{y}))\mathrm{d}y=\int_e^{+\infty}(1-\Phi(x))\mathrm{d}(x-e)^2=\int_e^{+\infty}(x-e)^2\mathrm{d}\Phi(x).$$

Similarly, substituting $e-\sqrt{y}$ with x and y with $(x-e)^2$, we obtain

$$\int_0^{+\infty}\Phi(e-\sqrt{y})\mathrm{d}y=\int_e^{-\infty}\Phi(x)\mathrm{d}(x-e)^2=\int_{-\infty}^{e}(x-e)^2\mathrm{d}\Phi(x).$$

It follows that the variance is

$$V[\xi]=\int_e^{+\infty}(x-e)^2\mathrm{d}\Phi(x)+\int_{-\infty}^{e}(x-e)^2\mathrm{d}\Phi(x)=\int_{-\infty}^{+\infty}(x-e)^2\mathrm{d}\Phi(x).$$

The theorem is verified.

Theorem B.25 *(Sheng and Yao [211]) Let ξ be an uncertain random variable with regular chance distribution Φ and finite expected value e. Then*

$$V[\xi]=\int_0^1(\Phi^{-1}(\alpha)-e)^2\mathrm{d}\alpha. \tag{B.82}$$

Proof: Substituting $\Phi(x)$ with α and x with $\Phi^{-1}(\alpha)$, it follows from the change of variables of integral and Theorem B.24 that the variance is

$$V[\xi]=\int_{-\infty}^{+\infty}(x-e)^2\mathrm{d}\Phi(x)=\int_0^1(\Phi^{-1}(\alpha)-e)^2\mathrm{d}\alpha.$$

The theorem is verified.

Theorem B.26 *(Guo and Wang [57]) Let $\eta_1,\eta_2,\cdots,\eta_m$ be independent random variables with probability distributions $\Psi_1,\Psi_2,\cdots,\Psi_m$, and let $\tau_1,\tau_2,\cdots,\tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1,\Upsilon_2,\cdots,\Upsilon_n$, respectively. Then*

$$\xi=f(\eta_1,\eta_2,\cdots,\eta_m,\tau_1,\tau_2,\cdots,\tau_n) \tag{B.83}$$

has a variance

$$\begin{aligned}V[\xi]\ =&\int_{\Re^m}\int_0^{+\infty}(1-F(e+\sqrt{x};y_1,\cdots,y_m)\\&+F(e-\sqrt{x};y_1,\cdots,y_m))\mathrm{d}x\mathrm{d}\Psi_1(y_1)\cdots\Psi_m(y_m)\end{aligned} \tag{B.84}$$

where $F(x;y_1,\cdots,y_m)$ is the uncertainty distribution of the uncertain variable $f(y_1,\cdots,y_m,\tau_1,\cdots,\tau_n)$ and is determined by $\Upsilon_1,\Upsilon_2,\cdots,\Upsilon_n$.

Proof: It follows from the operational law of uncertain random variables that ξ has a chance distribution

$$\Phi(x) = \int_{\Re^m} F(x; y_1, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \cdots \Psi_m(y_m).$$

Thus the theorem follows Stipulation B.1 immediately.

Exercise B.11: Let η be a random variable with probability distribution Ψ, and let τ be an uncertain variable with uncertainty distribution Υ. Show that the sum

$$\xi = \eta + \tau \tag{B.85}$$

has a variance

$$V[\xi] = \int_{-\infty}^{+\infty} \int_0^{+\infty} (1 - \Upsilon(e + \sqrt{x} - y) + \Upsilon(e - \sqrt{x} - y)) \mathrm{d}x \mathrm{d}\Psi(y). \tag{B.86}$$

B.7 Law of Large Numbers

Theorem B.27 *(Yao and Gao [251], Law of Large Numbers) Let $\eta_1, \eta_2, \cdots$ be iid random variables with a common probability distribution Ψ, and let $\tau_1, \tau_2, \cdots$ be iid uncertain variables. Assume f is a strictly monotone function. Then*

$$S_n = f(\eta_1, \tau_1) + f(\eta_2, \tau_2) + \cdots + f(\eta_n, \tau_n) \tag{B.87}$$

is a sequence of uncertain random variables and

$$\frac{S_n}{n} \to \int_{-\infty}^{+\infty} f(y, \tau_1) \mathrm{d}\Psi(y) \tag{B.88}$$

in the sense of convergence in distribution as $n \to \infty$.

Proof: According to the definition of convergence in distribution, it suffices to prove

$$\begin{aligned} &\lim_{n \to \infty} \mathrm{Ch}\left\{ \frac{S_n}{n} \le \int_{-\infty}^{+\infty} f(y, z) \mathrm{d}\Psi(y) \right\} \\ &= \mathcal{M}\left\{ \int_{-\infty}^{+\infty} f(y, \tau_1) \mathrm{d}\Psi(y) \le \int_{-\infty}^{+\infty} f(y, z) \mathrm{d}\Psi(y) \right\} \end{aligned} \tag{B.89}$$

for any real number z with

$$\begin{aligned} &\lim_{w \to z} \mathcal{M}\left\{ \int_{-\infty}^{+\infty} f(y, \tau_1) \mathrm{d}\Psi(y) \le \int_{-\infty}^{+\infty} f(y, w) \mathrm{d}\Psi(y) \right\} \\ &= \mathcal{M}\left\{ \int_{-\infty}^{+\infty} f(y, \tau_1) \mathrm{d}\Psi(y) \le \int_{-\infty}^{+\infty} f(y, z) \mathrm{d}\Psi(y) \right\}. \end{aligned}$$

The argument breaks into two cases. Case 1: Assume $f(y,z)$ is strictly increasing with respect to z. Let Υ denote the common uncertainty distribution of $\tau_1,\tau_2,\cdots$ It is clear that

$$\mathcal{M}\{f(y,\tau_1)\le f(y,z)\}=\Upsilon(z)$$

for any real numbers y and z. Thus we have

$$\mathcal{M}\left\{\int_{-\infty}^{+\infty}f(y,\tau_1)\mathrm{d}\Psi(y)\le\int_{-\infty}^{+\infty}f(y,z)\mathrm{d}\Psi(y)\right\}=\Upsilon(z). \tag{B.90}$$

In addition, since $f(\eta_1,z),f(\eta_2,z),\cdots$ are a sequence of iid random variables, the law of large numbers for random variables tells us that

$$\frac{f(\eta_1,z)+f(\eta_2,z)+\cdots+f(\eta_n,z)}{n}\to\int_{-\infty}^{+\infty}f(y,z)\mathrm{d}\Psi(y),\quad a.s.$$

as $n\to\infty$. Thus

$$\lim_{n\to\infty}\mathrm{Ch}\left\{\frac{S_n}{n}\le\int_{-\infty}^{+\infty}f(y,z)\mathrm{d}\Psi(y)\right\}=\Upsilon(z). \tag{B.91}$$

It follows from (B.90) and (B.91) that (B.89) holds. Case 2: Assume $f(y,z)$ is strictly decreasing with respect to z. Then $-f(y,z)$ is strictly increasing with respect to z. By using Case 1 we obtain

$$\lim_{n\to\infty}\mathrm{Ch}\left\{-\frac{S_n}{n}<-z\right\}=\mathcal{M}\left\{-\int_{-\infty}^{+\infty}f(y,\tau_1)\mathrm{d}\Psi(y)<-z\right\}.$$

That is,

$$\lim_{n\to\infty}\mathrm{Ch}\left\{\frac{S_n}{n}>z\right\}=\mathcal{M}\left\{\int_{-\infty}^{+\infty}f(y,\tau_1)\mathrm{d}\Psi(y)>z\right\}.$$

It follows from the duality property that

$$\lim_{n\to\infty}\mathrm{Ch}\left\{\frac{S_n}{n}\le z\right\}=\mathcal{M}\left\{\int_{-\infty}^{+\infty}f(y,\tau_1)\mathrm{d}\Psi(y)\le z\right\}.$$

The theorem is thus proved.

Exercise B.12: Let $\eta_1,\eta_2,\cdots$ be iid random variables, and let $\tau_1,\tau_2,\cdots$ be iid uncertain variables. Define

$$S_n=(\eta_1+\tau_1)+(\eta_2+\tau_2)+\cdots+(\eta_n+\tau_n). \tag{B.92}$$

Show that

$$\frac{S_n}{n}\to E[\eta_1]+\tau_1 \tag{B.93}$$

in the sense of convergence in distribution as $n \to \infty$.

Exercise B.13: Let $\eta_1, \eta_2, \cdots$ be iid positive random variables, and let $\tau_1, \tau_2, \cdots$ be iid positive uncertain variables. Define

$$S_n = \eta_1\tau_1 + \eta_2\tau_2 + \cdots + \eta_n\tau_n. \tag{B.94}$$

Show that

$$\frac{S_n}{n} \to E[\eta_1]\tau_1 \tag{B.95}$$

in the sense of convergence in distribution as $n \to \infty$.

B.8 Uncertain Random Programming

Assume that $\boldsymbol{x}$ is a decision vector, and $\boldsymbol{\xi}$ is an uncertain random vector. Since an uncertain random objective function $f(\boldsymbol{x}, \boldsymbol{\xi})$ cannot be directly minimized, we may minimize its expected value, i.e.,

$$\min_{\boldsymbol{x}} E[f(\boldsymbol{x}, \boldsymbol{\xi})]. \tag{B.96}$$

Since the uncertain random constraints $g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0, j = 1, 2, \cdots, p$ do not make a crisp feasible set, it is naturally desired that the uncertain random constraints hold with confidence levels $\alpha_1, \alpha_2, \cdots, \alpha_p$. Then we have a set of chance constraints,

$$\text{Ch}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \cdots, p. \tag{B.97}$$

In order to obtain a decision with minimum expected objective value subject to a set of chance constraints, Liu [150] proposed the following uncertain random programming model,

$$\begin{cases} \min\limits_{\boldsymbol{x}} E[f(\boldsymbol{x}, \boldsymbol{\xi})] \\ \text{subject to:} \\ \qquad \text{Ch}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\} \geq \alpha_j, \quad j = 1, 2, \cdots, p. \end{cases} \tag{B.98}$$

Definition B.6 *(Liu [150]) A vector $\boldsymbol{x}$ is called a feasible solution to the uncertain random programming model (B.98) if*

$$\text{Ch}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\} \geq \alpha_j \tag{B.99}$$

for $j = 1, 2, \cdots, p$.

Definition B.7 *(Liu [150]) A feasible solution $\boldsymbol{x}^*$ is called an optimal solution to the uncertain random programming model (B.98) if*

$$E[f(\boldsymbol{x}^*, \boldsymbol{\xi})] \leq E[f(\boldsymbol{x}, \boldsymbol{\xi})] \tag{B.100}$$

for any feasible solution $\boldsymbol{x}$.

Theorem B.28 *(Liu [150]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If $f(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ is a strictly increasing function or a strictly decreasing function with respect to $\tau_1, \cdots, \tau_n$, then the expected function*

$$E[f(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)] \tag{B.101}$$

is equal to

$$\int_{\Re^m} \int_0^1 f(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m).$$

Proof: It follows from Theorem B.20 immediately.

Remark B.13: If $f(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ is strictly increasing with respect to $\tau_1, \cdots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \cdots, \tau_n$, then the integrand in the formula of expected value $E[f(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)]$ should be replaced with

$$f(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)).$$

Theorem B.29 *(Liu [150]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If $g_j(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ is a strictly increasing function with respect to $\tau_1, \cdots, \tau_n$, then the chance constraint*

$$\mathrm{Ch}\{g_j(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \le 0\} \ge \alpha_j \tag{B.102}$$

holds if and only if

$$\int_{\Re^m} G_j(\boldsymbol{x}, y_1, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \ge \alpha_j \tag{B.103}$$

where $G_j(\boldsymbol{x}, y_1, \cdots, y_m)$ is the root α of the equation

$$g_j(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) = 0. \tag{B.104}$$

Proof: Since $G_j(\boldsymbol{x}, y_1, \cdots, y_m)$ is the root α of the equation (B.104), it follows from Theorem B.13 that the chance measure

$$\mathrm{Ch}\{g_j(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \le 0\}$$

is equal to the integral

$$\int_{\Re^m} G_j(\boldsymbol{x}, y_1, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m).$$

Hence the chance constraint (B.102) holds if and only if (B.103) is true. The theorem is verified.

Remark B.14: Sometimes, the equation (B.104) may not have a root. In this case, if

$$g_j(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) < 0 \tag{B.105}$$

for all α, then we set the root $\alpha = 1$; and if

$$g_j(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) > 0 \tag{B.106}$$

for all α, then we set the root $\alpha = 0$.

Remark B.15: The root α may be estimated by the bisection method because $g_j(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha))$ is a strictly increasing function with respect to α.

Remark B.16: If $g_j(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ is strictly increasing with respect to $\tau_1, \cdots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \cdots, \tau_n$, then the equation (B.104) becomes

$$g_j(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_k^{-1}(\alpha), \Upsilon_{k+1}^{-1}(1-\alpha), \cdots, \Upsilon_n^{-1}(1-\alpha)) = 0.$$

Theorem B.30 *(Liu [150]) Let $\eta_1, \eta_2, \cdots, \eta_m$ be independent random variables with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$, and let $\tau_1, \tau_2, \cdots, \tau_n$ be independent uncertain variables with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If $f(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ and $g_j(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ are strictly increasing functions with respect to $\tau_1, \cdots, \tau_n$ for $j = 1, 2, \cdots, p$, then the uncertain random programming*

$$\begin{cases} \min\limits_{\boldsymbol{x}} E[f(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)] \\ \text{subject to:} \\ \qquad \mathrm{Ch}\{g_j(\boldsymbol{x}, \eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n) \le 0\} \ge \alpha_j, \ j = 1, 2, \cdots, p \end{cases}$$

is equivalent to the crisp mathematical programming

$$\begin{cases} \min\limits_{\boldsymbol{x}} \displaystyle\int_{\Re^m} \int_0^1 f(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) \mathrm{d}\alpha \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \\ \text{subject to:} \\ \qquad \displaystyle\int_{\Re^m} G_j(\boldsymbol{x}, y_1, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \ge \alpha_j, \ j = 1, 2, \cdots, p \end{cases}$$

where $G_j(\boldsymbol{x}, y_1, \cdots, y_m)$ are the roots α of the equations

$$g_j(\boldsymbol{x}, y_1, \cdots, y_m, \Upsilon_1^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) = 0 \tag{B.107}$$

for $j = 1, 2, \cdots, p$, respectively.

Proof: It follows from Theorems B.28 and B.29 immediately.

After an uncertain random programming is converted into a crisp mathematical programming, we may solve it by any classical numerical methods (e.g. iterative method) or intelligent algorithms (e.g. genetic algorithm).

B.9 Uncertain Random Risk Analysis

The study of uncertain random risk analysis was started by Liu and Ralescu [151] with the concept of risk index.

Definition B.8 *(Liu and Ralescu [151]) Assume that a system contains uncertain random factors $\xi_1, \xi_2, \cdots, \xi_n$, and has a loss function f. Then the risk index is the chance measure that the system is loss-positive, i.e.,*

$$Risk = \text{Ch}\{f(\xi_1, \xi_2, \cdots, \xi_n) > 0\}. \tag{B.108}$$

If all uncertain random factors degenerate to random ones, then the risk index is the probability measure that the system is loss-positive (Roy [199]). If all uncertain random factors degenerate to uncertain ones, then the risk index is the uncertain measure that the system is loss-positive (Liu [128]).

Theorem B.31 *(Liu and Ralescu [151], Risk Index Theorem) Assume a system contains independent random variables $\eta_1, \eta_2, \cdots, \eta_m$ with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$ and independent uncertain variables $\tau_1, \tau_2, \cdots, \tau_n$ with regular uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If the loss function $f(\eta_1, \cdots, \eta_m, \tau_1, \cdots, \tau_n)$ is strictly increasing with respect to $\tau_1, \cdots, \tau_k$ and strictly decreasing with respect to $\tau_{k+1}, \cdots, \tau_n$, then the risk index is*

$$Risk = \int_{\Re^m} G(y_1, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \cdots \mathrm{d}\Psi_m(y_m) \tag{B.109}$$

where $G(y_1, \cdots, y_m)$ is the root α of the equation

$$f(y_1, \cdots, y_m, \Upsilon_1^{-1}(1-\alpha), \cdots, \Upsilon_k^{-1}(1-\alpha), \Upsilon_{k+1}^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) = 0.$$

Proof: It follows from Definition B.8 and Theorem B.14 immediately.

Remark B.17: Sometimes, the equation may not have a root. In this case, if

$$f(y_1, \cdots, y_m, \Upsilon_1^{-1}(1-\alpha), \cdots, \Upsilon_k^{-1}(1-\alpha), \Upsilon_{k+1}^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) < 0$$

for all α, then we set the root $\alpha = 0$; and if

$$f(y_1, \cdots, y_m, \Upsilon_1^{-1}(1-\alpha), \cdots, \Upsilon_k^{-1}(1-\alpha), \Upsilon_{k+1}^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha)) > 0$$

for all α, then we set the root $\alpha = 1$.

Remark B.18: The root α may be estimated by the bisection method because $f(y_1, \cdots, y_m, \Upsilon_1^{-1}(1-\alpha), \cdots, \Upsilon_k^{-1}(1-\alpha), \Upsilon_{k+1}^{-1}(\alpha), \cdots, \Upsilon_n^{-1}(\alpha))$ is a strictly decreasing function with respect to α.

Exercise B.14: (Series System) Consider a series system in which there are m elements whose lifetimes are independent random variables $\eta_1, \eta_2, \cdots, \eta_m$ with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$ and n elements whose lifetimes are independent uncertain variables $\tau_1, \tau_2, \cdots, \tau_n$ with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f = T - \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_m \wedge \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n. \tag{B.110}$$

Show that the risk index is

$$Risk = a + b - ab \tag{B.111}$$

where

$$a = 1 - (1 - \Psi_1(T))(1 - \Psi_2(T)) \cdots (1 - \Psi_m(T)), \tag{B.112}$$

$$b = \Upsilon_1(T) \vee \Upsilon_2(T) \vee \cdots \vee \Upsilon_n(T). \tag{B.113}$$

Exercise B.15: (Parallel System) Consider a parallel system in which there are m elements whose lifetimes are independent random variables $\eta_1, \eta_2, \cdots,$ η_m with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$ and n elements whose lifetimes are independent uncertain variables $\tau_1, \tau_2, \cdots, \tau_n$ with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f = T - \eta_1 \vee \eta_2 \vee \cdots \vee \eta_m \vee \tau_1 \vee \tau_2 \vee \cdots \vee \tau_n. \tag{B.114}$$

Show that the risk index is

$$Risk = ab \tag{B.115}$$

where

$$a = \Psi_1(T)\Psi_2(T) \cdots \Psi_m(T), \tag{B.116}$$

$$b = \Upsilon_1(T) \wedge \Upsilon_2(T) \wedge \cdots \wedge \Upsilon_n(T). \tag{B.117}$$

Exercise B.16: (Standby System) Consider a standby system in which there are m elements whose lifetimes are independent random variables $\eta_1, \eta_2,$ $\cdots, \eta_m$ with probability distributions $\Psi_1, \Psi_2, \cdots, \Psi_m$ and n elements whose lifetimes are independent uncertain variables $\tau_1, \tau_2, \cdots, \tau_n$ with uncertainty distributions $\Upsilon_1, \Upsilon_2, \cdots, \Upsilon_n$, respectively. If the loss is understood as the case that the system fails before the time T, then the loss function is

$$f = T - (\eta_1 + \eta_2 + \cdots + \eta_m + \tau_1 + \tau_2 + \cdots + \tau_n). \tag{B.118}$$

Show that the risk index is

$$Risk = \int_{\Re^m} G(y_1, y_2, \cdots, y_m) \mathrm{d}\Psi_1(y_1) \mathrm{d}\Psi_2(y_2) \cdots \mathrm{d}\Psi_m(y_m) \tag{B.119}$$

where $G(y_1, y_2, \cdots, y_m)$ is the root α of the equation

$$\Upsilon_1^{-1}(\alpha) + \Upsilon_2^{-1}(\alpha) + \cdots + \Upsilon_n^{-1}(\alpha) = T - (y_1 + y_2 + \cdots + y_m). \tag{B.120}$$

Remark B.19: As a substitute of risk index, Liu and Ralescu [153] suggested a concept of value-at-risk,

$$\mathrm{VaR}(\alpha) = \sup\{x \mid \mathrm{Ch}\{f(\xi_1, \xi_2, \cdots, \xi_n) \geq x\} \geq \alpha\}. \tag{B.121}$$

Note that $\mathrm{VaR}(\alpha)$ represents the maximum possible loss when α percent of the right tail distribution is ignored. In other words, the loss will exceed $\mathrm{VaR}(\alpha)$ with chance measure α. If $\Phi(x)$ is the chance distribution of $f(\xi_1, \xi_2, \cdots, \xi_n)$, then

$$\mathrm{VaR}(\alpha) = \sup\left\{x \mid \Phi(x) \leq 1 - \alpha\right\}. \tag{B.122}$$

If its inverse uncertainty distribution $\Phi^{-1}(\alpha)$ exists, then

$$\mathrm{VaR}(\alpha) = \Phi^{-1}(1 - \alpha). \tag{B.123}$$

When the uncertain random variables degenerate to random variables, the value-at-risk becomes the one in Morgan [171]. When the uncertain random variables degenerate to uncertain variables, the value-at-risk becomes the one in Peng [183].

Remark B.20: Liu and Ralescu [151] proposed a concept of expected loss that is the expected value of the loss $f(\xi_1, \xi_2, \cdots, \xi_n)$ given $f(\xi_1, \xi_2, \cdots, \xi_n) > 0$. That is,

$$L = \int_0^{+\infty} \mathcal{M}\{f(\xi_1, \xi_2, \cdots, \xi_n) \geq x\} \mathrm{d}x. \tag{B.124}$$

If $\Phi(x)$ is the chance distribution of the loss $f(\xi_1, \xi_2, \cdots, \xi_n)$, then we immediately have

$$L = \int_0^{+\infty} (1 - \Phi(x)) \mathrm{d}x. \tag{B.125}$$

If its inverse uncertainty distribution $\Phi^{-1}(\alpha)$ exists, then the expected loss is

$$L = \int_0^1 \left(\Phi^{-1}(\alpha)\right)^+ \mathrm{d}\alpha. \tag{B.126}$$

B.10 Uncertain Random Reliability Analysis

The study of uncertain random reliability analysis was started by Wen and Kang [232] with the concept of reliability index.

Definition B.9 *(Wen and Kang [232]) Assume a Boolean system has uncertain random elements $\xi_1, \xi_2, \cdots, \xi_n$ and a structure function f. Then the reliability index is the chance measure that the system is working, i.e.,*

$$\textit{Reliability} = \text{Ch}\{f(\xi_1, \xi_2, \cdots, \xi_n) = 1\}. \tag{B.127}$$

If all uncertain random elements degenerate to random ones, then the reliability index is the probability measure that the system is working. If all uncertain random elements degenerate to uncertain ones, then the reliability index (Liu [128]) is the uncertain measure that the system is working.

Theorem B.32 *(Wen and Kang [232], Reliability Index Theorem) Assume that a system has a structure function f and contains independent random elements $\eta_1, \eta_2, \cdots, \eta_m$ with reliabilities $a_1, a_2, \cdots, a_m$, and independent uncertain elements $\tau_1, \tau_2, \cdots, \tau_n$ with reliabilities $b_1, b_2, \cdots, b_n$, respectively. Then the reliability index is*

$$\textit{Reliability} = \sum_{(x_1,\cdots,x_m)\in\{0,1\}^m} \left(\prod_{i=1}^m \mu_i(x_i)\right) f^*(x_1, \cdots, x_m) \tag{B.128}$$

where

$$f^*(x_1,\cdots,x_m) = \begin{cases} \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j), \\ \quad \text{if } \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j) < 0.5 \\ 1 - \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=0} \min\limits_{1\le j\le n} \nu_j(y_j), \\ \quad \text{if } \sup\limits_{f(x_1,\cdots,x_m,y_1,\cdots,y_n)=1} \min\limits_{1\le j\le n} \nu_j(y_j) \ge 0.5, \end{cases} \tag{B.129}$$

$$\mu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \quad (i = 1, 2, \cdots, m), \tag{B.130}$$

$$\nu_j(y_j) = \begin{cases} b_j, & \text{if } y_j = 1 \\ 1 - b_j, & \text{if } y_j = 0 \end{cases} \quad (j = 1, 2, \cdots, n). \tag{B.131}$$

Proof: It follows from Definition B.9 and Theorem B.15 immediately.

Exercise B.17: (Series System) Consider a series system in which there are m independent random elements $\eta_1, \eta_2, \cdots, \eta_m$ with reliabilities $a_1, a_2, \cdots, a_m$, and n independent uncertain elements $\tau_1, \tau_2, \cdots, \tau_n$ with reliabilities $b_1, b_2, \cdots,$ b_n, respectively. Note that the structure function is

$$f = \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_m \wedge \tau_1 \wedge \tau_2 \wedge \cdots \wedge \tau_n. \tag{B.132}$$

Show that the reliability index is

$$\textit{Reliability} = a_1 a_2 \cdots a_m (b_1 \wedge b_2 \wedge \cdots \wedge b_n). \tag{B.133}$$

Exercise B.18: (Parallel System) Consider a parallel system in which there are m independent random elements $\eta_1, \eta_2, \cdots, \eta_m$ with reliabilities $a_1, a_2, \cdots, a_m$, and n independent uncertain elements $\tau_1, \tau_2, \cdots, \tau_n$ with reliabilities $b_1, b_2, \cdots, b_n$, respectively. Note that the structure function is

$$f = \eta_1 \vee \eta_2 \vee \cdots \vee \eta_m \vee \tau_1 \vee \tau_2 \vee \cdots \vee \tau_n. \tag{B.134}$$

Show that the reliability index is

$$\textit{Reliability} = 1 - (1 - a_1)(1 - a_2) \cdots (1 - a_m)(1 - b_1 \vee b_2 \vee \cdots \vee b_n). \tag{B.135}$$

Exercise B.19: (k-out-of-$(m+n)$ System) Consider a k-out-of-$(m+n)$ system in which there are m independent random elements $\eta_1, \eta_2, \cdots, \eta_m$ with reliabilities $a_1, a_2, \cdots, a_m$, and n independent uncertain elements $\tau_1, \tau_2, \cdots, \tau_n$ with reliabilities $b_1, b_2, \cdots, b_n$, respectively. Note that the structure function is

$$f = k\text{-max}\,[\eta_1, \eta_2, \cdots, \eta_m, \tau_1, \tau_2, \cdots, \tau_n]. \tag{B.136}$$

Show that the reliability index is

$$\textit{Reliability} = \sum_{(x_1, x_2, \cdots, x_m) \in \{0,1\}^m} \left(\prod_{i=1}^{m} \mu_i(x_i) \right) f^*(x_1, x_2, \cdots, x_m) \tag{B.137}$$

where

$$f^*(x_1, x_2, \cdots, x_m) = k\text{-max}\,[x_1, x_2, \cdots, x_m, b_1, b_2, \cdots, b_n], \tag{B.138}$$

$$\mu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases} \quad (i = 1, 2, \cdots, m). \tag{B.139}$$

B.11 Uncertain Random Graph

In classic graph theory, the edges and vertices are all deterministic, either exist or not. However, in practical applications, some indeterminate factors will no doubt appear in graphs. Thus it is reasonable to assume that in a graph some edges exist with some degrees in probability measure and others exist with some degrees in uncertain measure. In order to model this type of graph, Liu [138] presented a concept of uncertain random graph.

We say a graph is of order n if it has n vertices labeled by $1, 2, \cdots, n$. In this section, we assume the graph is always of order n, and has a collection of vertices,

$$\mathcal{V} = \{1, 2, \cdots, n\}. \tag{B.140}$$

Let us define two collections of edges,

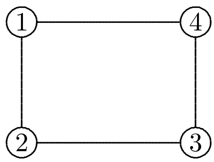
$$\mathcal{U} = \{(i, j) \mid 1 \le i < j \le n \text{ and } (i, j) \text{ are uncertain edges}\}, \tag{B.141}$$

$$\mathcal{R} = \{(i, j) \mid 1 \le i < j \le n \text{ and } (i, j) \text{ are random edges}\}. \tag{B.142}$$

Note that all deterministic edges are regarded as special uncertain ones. Then $\mathcal{U} \cup \mathcal{R} = \{(i, j) \mid 1 \le i < j \le n\}$ that contains $n(n-1)/2$ edges. We will call

$$\mathcal{T} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix} \tag{B.143}$$

an *uncertain random adjacency matrix* if α_{ij} represent the truth values in uncertain measure or probability measure that the edges between vertices i and j exist, $i, j = 1, 2, \cdots, n$, respectively. Note that $\alpha_{ii} = 0$ for $i = 1, 2, \cdots, n$, and $\mathcal{T}$ is a symmetric matrix, i.e., $\alpha_{ij} = \alpha_{ji}$ for $i, j = 1, 2, \cdots, n$.



$$\begin{pmatrix} 0 & 0.8 & 0 & 0.5 \\ 0.8 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0.3 \\ 0.5 & 0 & 0.3 & 0 \end{pmatrix}$$

Figure B.4: An Uncertain Random Graph

Definition B.10 *(Liu [138]) Assume $\mathcal{V}$ is the collection of vertices, $\mathcal{U}$ is the collection of uncertain edges, $\mathcal{R}$ is the collection of random edges, and $\mathcal{T}$ is the uncertain random adjacency matrix. Then the quartette $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ is said to be an uncertain random graph.*

Please note that the uncertain random graph becomes a *random graph* (Erdős and Rényi [38], Gilbert [56]) if the collection $\mathcal{U}$ of uncertain edges vanishes; and becomes an *uncertain graph* (Gao and Gao [50]) if the collection $\mathcal{R}$ of random edges vanishes.

In order to deal with uncertain random graph, let us introduce some symbols. Write

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \tag{B.144}$$

and

$$\mathbb{X} = \left\{ X \mid \begin{array}{l} x_{ij} = 0 \text{ or } 1, \text{ if } (i,j) \in \mathcal{R} \\ x_{ij} = 0, \text{ if } (i,j) \in \mathcal{U} \\ x_{ij} = x_{ji}, \; i,j = 1,2,\cdots,n \\ x_{ii} = 0, \; i = 1,2,\cdots,n \end{array} \right\}. \tag{B.145}$$

For each given matrix

$$Y = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{pmatrix}, \tag{B.146}$$

the *extension class* of Y is defined by

$$Y^* = \left\{ X \mid \begin{array}{l} x_{ij} = y_{ij}, \text{ if } (i,j) \in \mathcal{R} \\ x_{ij} = 0 \text{ or } 1, \text{ if } (i,j) \in \mathcal{U} \\ x_{ij} = x_{ji}, \; i,j = 1,2,\cdots,n \\ x_{ii} = 0, \; i = 1,2,\cdots,n \end{array} \right\}. \tag{B.147}$$

Example B.6: (Liu [138], Connectivity Index) An uncertain random graph is connected for some realizations of uncertain and random edges, and disconnected for some other realizations. In order to show how likely an uncertain random graph is connected, a connectivity index of an uncertain random graph is defined as the chance measure that the uncertain random graph is connected. Let $(\mathcal{V}, \mathcal{U}, \mathcal{R}, \mathcal{T})$ be an uncertain random graph. Liu [138] proved that the connectivity index is

$$\rho = \sum_{Y \in \mathbb{X}} \left(\prod_{(i,j) \in \mathcal{R}} \nu_{ij}(Y) \right) f^*(Y) \tag{B.148}$$

where

$$f^*(Y) = \begin{cases} \displaystyle\sup_{X \in Y^*, f(X)=1} \min_{(i,j) \in \mathcal{U}} \nu_{ij}(X), & \text{if } \displaystyle\sup_{X \in Y^*, f(X)=1} \min_{(i,j) \in \mathcal{U}} \nu_{ij}(X) < 0.5 \\ 1 - \displaystyle\sup_{X \in Y^*, f(X)=0} \min_{(i,j) \in \mathcal{U}} \nu_{ij}(X), & \text{if } \displaystyle\sup_{X \in Y^*, f(X)=1} \min_{(i,j) \in \mathcal{U}} \nu_{ij}(X) \geq 0.5, \end{cases}$$

$$\nu_{ij}(X) = \begin{cases} \alpha_{ij}, & \text{if } x_{ij} = 1 \\ 1 - \alpha_{ij}, & \text{if } x_{ij} = 0 \end{cases} \qquad (i,j) \in \mathfrak{U}, \tag{B.149}$$

$$f(X) = \begin{cases} 1, & \text{if } I + X + X^2 + \cdots + X^{n-1} > 0 \\ 0, & \text{otherwise,} \end{cases} \tag{B.150}$$

$\mathbb{X}$ is the class of matrixes satisfying (B.145), and Y^* is the extension class of Y satisfying (B.147).

Remark B.21: If the uncertain random graph becomes a random graph, then the connectivity index is

$$\rho = \sum_{X \in \mathbb{X}} \left(\prod_{1 \le i < j \le n} \nu_{ij}(X) \right) f(X) \tag{B.151}$$

where

$$\mathbb{X} = \left\{ X \mid \begin{array}{c} x_{ij} = 0 \text{ or } 1,\ i,j = 1,2,\cdots,n \\ x_{ij} = x_{ji},\ i,j = 1,2,\cdots,n \\ x_{ii} = 0,\ i = 1,2,\cdots,n \end{array} \right\}. \tag{B.152}$$

Remark B.22: (Gao and Gao [50]) If the uncertain random graph becomes an uncertain graph, then the connectivity index is

$$\rho = \begin{cases} \sup\limits_{X \in \mathbb{X}, f(X)=1} \min\limits_{1 \le i < j \le n} \nu_{ij}(X), & \text{if } \sup\limits_{X \in \mathbb{X}, f(X)=1} \min\limits_{1 \le i < j \le n} \nu_{ij}(X) < 0.5 \\ 1 - \sup\limits_{X \in \mathbb{X}, f(X)=0} \min\limits_{1 \le i < j \le n} \nu_{ij}(X), & \text{if } \sup\limits_{X \in \mathbb{X}, f(X)=1} \min\limits_{1 \le i < j \le n} \nu_{ij}(X) \ge 0.5 \end{cases}$$

where $\mathbb{X}$ becomes

$$\mathbb{X} = \left\{ X \mid \begin{array}{c} x_{ij} = 0 \text{ or } 1,\ i,j = 1,2,\cdots,n \\ x_{ij} = x_{ji},\ i,j = 1,2,\cdots,n \\ x_{ii} = 0,\ i = 1,2,\cdots,n \end{array} \right\}. \tag{B.153}$$

Exercise B.20: An Euler circuit in the graph is a circuit that passes through each edge exactly once. In other words, a graph has an Euler circuit if it can be drawn on paper without ever lifting the pencil and without retracing over any edge. It has been proved that a graph has an Euler circuit if and only if it is connected and each vertex has an even degree (i.e., the number of edges that are adjacent to that vertex). In order to measure how likely an uncertain random graph has an Euler circuit, an Euler index is defined as the chance measure that the uncertain random graph has an Euler circuit. Please give a formula for calculating Euler index.

B.12 Uncertain Random Network

The term *network* is a synonym for a weighted graph, where the weights may be understood as cost, distance or time consumed. Assume that in a network some weights are random variables and others are uncertain variables. In order to model this type of network, Liu [138] presented a concept of uncertain random network.

In this section, we assume the uncertain random network is always of order n, and has a collection of nodes,

$$\mathcal{N} = \{1, 2, \cdots, n\} \tag{B.154}$$

where "1" is always the source node, and "n" is always the destination node. Let us define two collections of arcs,

$$\mathcal{U} = \{(i,j) \mid (i,j) \text{ are uncertain arcs}\}, \tag{B.155}$$

$$\mathcal{R} = \{(i,j) \mid (i,j) \text{ are random arcs}\}. \tag{B.156}$$

Note that all deterministic arcs are regarded as special uncertain ones. Let w_{ij} denote the weights of arcs (i,j), $(i,j) \in \mathcal{U} \cup \mathcal{R}$, respectively. Then w_{ij} are uncertain variables if $(i,j) \in \mathcal{U}$, and random variables if $(i,j) \in \mathcal{R}$. Write

$$\mathcal{W} = \{w_{ij} \mid (i,j) \in \mathcal{U} \cup \mathcal{R}\}. \tag{B.157}$$

Definition B.11 *(Liu [138]) Assume* $\mathcal{N}$ *is the collection of nodes,* $\mathcal{U}$ *is the collection of uncertain arcs,* $\mathcal{R}$ *is the collection of random arcs, and* $\mathcal{W}$ *is the collection of uncertain and random weights. Then the quartette* $(\mathcal{N}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ *is said to be an uncertain random network.*

Please note that the uncertain random network becomes a *random network* (Frank and Hakimi [43]) if all weights are random variables; and becomes an *uncertain network* (Liu [129]) if all weights are uncertain variables.

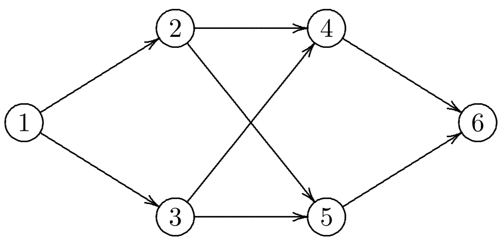


Figure B.5: An Uncertain Random Network

Figure B.5 shows an uncertain random network $(\mathcal{N}, \mathcal{U}, \mathcal{R}, \mathcal{W})$ of order 6 in which

$$\mathcal{N} = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}, \tag{B.158}$$

$$\mathcal{U} = \{(1,2),\ (1,3),\ (2,4),\ (2,5),\ (3,4),\ (3,5)\}, \tag{B.159}$$

$$\mathcal{R} = \{(4,6),\ (5,6)\}, \tag{B.160}$$

$$\mathcal{W} = \{w_{12},\ w_{13},\ w_{24},\ w_{25},\ w_{34},\ w_{35},\ w_{46},\ w_{56}\}. \tag{B.161}$$

Example B.7: (Liu [138], Shortest Path Distribution) Consider an uncertain random network $(\mathcal{N}, \mathcal{U}, \mathcal{R}, \mathcal{W})$. Assume the uncertain weights w_{ij} have uncertainty distributions Υ_{ij} for $(i,j) \in \mathcal{U}$, and the random weights w_{ij} have probability distributions Ψ_{ij} for $(i,j) \in \mathcal{R}$, respectively. Then the shortest path distribution from a source node to a destination node is

$$\Phi(x) = \int_0^{+\infty} \cdots \int_0^{+\infty} F(x; y_{ij}, (i,j) \in \mathcal{R}) \prod_{(i,j)\in\mathcal{R}} \mathrm{d}\Psi_{ij}(y_{ij}) \tag{B.162}$$

where $F(x; y_{ij}, (i,j) \in \mathcal{R})$ is determined by its inverse uncertainty distribution

$$F^{-1}(\alpha; y_{ij}, (i,j) \in \mathcal{R}) = f(c_{ij}, (i,j) \in \mathcal{U} \cup \mathcal{R}), \tag{B.163}$$

$$c_{ij} = \begin{cases} \Upsilon_{ij}^{-1}(\alpha), & \text{if } (i,j) \in \mathcal{U} \\ y_{ij}, & \text{if } (i,j) \in \mathcal{R}, \end{cases} \tag{B.164}$$

and f may be calculated by the Dijkstra algorithm (Dijkstra [34]) for each given α.

Remark B.23: If the uncertain random network becomes a random network, then the shortest path distribution is

$$\Phi(x) = \int_{f(y_{ij}, (i,j)\in\mathcal{R}) \le x} \prod_{(i,j)\in\mathcal{R}} \mathrm{d}\Psi_{ij}(y_{ij}). \tag{B.165}$$

Remark B.24: (Gao [51]) If the uncertain random network becomes an uncertain network, then the inverse shortest path distribution is

$$\Phi^{-1}(\alpha) = f(\Upsilon_{ij}^{-1}(\alpha), (i,j) \in \mathcal{U}). \tag{B.166}$$

Exercise B.21: (Sheng and Gao [212]) Maximum flow problem is to find a flow with maximum value from a source node to a destination node in an uncertain random network. What is the maximum flow distribution?

B.13 Uncertain Random Process

Uncertain random process is a sequence of uncertain random variables indexed by time. A formal definition is given below.

Definition B.12 *(Gao and Yao [47]) Let* $(\Gamma,\mathcal{L},\mathcal{M})\times(\Omega,\mathcal{A},\Pr)$ *be a chance space and let* T *be a totally ordered set (e.g. time). An uncertain random process is a function* $X_t(\gamma,\omega)$ *from* $T\times(\Gamma,\mathcal{L},\mathcal{M})\times(\Omega,\mathcal{A},\Pr)$ *to the set of real numbers such that* $\{X_t\in B\}$ *is an event in* $\mathcal{L}\times\mathcal{A}$ *for any Borel set* B *at each time* t*.*

Example B.8: A stochastic process is a sequence of random variables indexed by time, and then is a special type of uncertain random process.

Example B.9: An uncertain process is a sequence of uncertain variables indexed by time, and then is a special type of uncertain random process.

Example B.10: Let Y_t be a stochastic process, and let Z_t be an uncertain process. If f is a measurable function, then

$$X_t=f(Y_t,Z_t) \tag{B.167}$$

is an uncertain random process.

Definition B.13 *(Gao and Yao [47]) Let* $\eta_1,\eta_2,\cdots$ *be iid random variables, let* $\tau_1,\tau_2,\cdots$ *be iid uncertain variables, and let* f *be a positive and strictly monotone function. Define* $S_0=0$ *and*

$$S_n=f(\eta_1,\tau_1)+f(\eta_2,\tau_2)+\cdots+f(\eta_n,\tau_n) \tag{B.168}$$

for $n\geq 1$*. Then*

$$N_t=\max_{n\geq 0}\left\{n \,\middle|\, S_n\leq t\right\} \tag{B.169}$$

is called an uncertain random renewal process with interarrival times $f(\eta_1,\tau_1)$*,* $f(\eta_2,\tau_2),\cdots$

Theorem B.33 *(Gao and Yao [47]) Let* $\eta_1,\eta_2,\cdots$ *be iid random variables with a common probability distribution* Ψ*, let* $\tau_1,\tau_2,\cdots$ *be iid uncertain variables, and let* f *be a positive and strictly monotone function. Assume* N_t *is an uncertain random renewal process with interarrival times* $f(\eta_1,\tau_1),f(\eta_2,\tau_2),$ $\cdots$ *Then the average renewal number*

$$\frac{N_t}{t}\to\left(\int_{-\infty}^{+\infty}f(y,\tau_1)\mathrm{d}\Psi(y)\right)^{-1} \tag{B.170}$$

in the sense of convergence in distribution as $t\to\infty$*.*

Proof: Write $S_n=f(\eta_1,\tau_1)+f(\eta_2,\tau_2)+\cdots+f(\eta_n,\tau_n)$ for all $n\geq 1$. Let x be a continuous point of the uncertainty distribution of

$$\left(\int_{-\infty}^{+\infty}f(y,\tau_1)\mathrm{d}\Psi(y)\right)^{-1}.$$

It is clear that $1/x$ is a continuous point of the uncertainty distribution of

$$\int_{-\infty}^{+\infty} f(y,\tau_1)\mathrm{d}\Psi(y).$$

At first, it follows from the definition of uncertain random renewal process that

$$\mathrm{Ch}\left\{\frac{N_t}{t}\le x\right\}=\mathrm{Ch}\left\{S_{\lfloor tx\rfloor+1}>t\right\}=\mathrm{Ch}\left\{\frac{S_{\lfloor tx\rfloor+1}}{\lfloor tx\rfloor+1}>\frac{t}{\lfloor tx\rfloor+1}\right\}$$

where $\lfloor tx\rfloor$ represents the maximal integer less than or equal to tx. Since $\lfloor tx\rfloor\le tx<\lfloor tx\rfloor+1$, we immediately have

$$\frac{\lfloor tx\rfloor}{\lfloor tx\rfloor+1}\cdot\frac{1}{x}\le\frac{t}{\lfloor tx\rfloor+1}<\frac{1}{x}$$

and then

$$\mathrm{Ch}\left\{\frac{S_{\lfloor tx\rfloor+1}}{\lfloor tx\rfloor+1}>\frac{1}{x}\right\}\le\mathrm{Ch}\left\{\frac{S_{\lfloor tx\rfloor+1}}{\lfloor tx\rfloor+1}>\frac{t}{\lfloor tx\rfloor+1}\right\}\le\mathrm{Ch}\left\{\frac{S_{\lfloor tx\rfloor+1}}{\lfloor tx\rfloor}>\frac{1}{x}\right\}.$$

It follows from the law of large numbers for uncertain random variables that

$$\begin{aligned}\lim_{t\to\infty}\mathrm{Ch}\left\{\frac{S_{\lfloor tx\rfloor+1}}{\lfloor tx\rfloor+1}>\frac{1}{x}\right\}&=1-\lim_{t\to\infty}\mathrm{Ch}\left\{\frac{S_{\lfloor tx\rfloor+1}}{\lfloor tx\rfloor+1}\le\frac{1}{x}\right\}\\&=1-\mathcal{M}\left\{\int_{-\infty}^{+\infty}f(y,\tau_1)\mathrm{d}\Psi(y)\le\frac{1}{x}\right\}\\&=\mathcal{M}\left\{\left(\int_{-\infty}^{+\infty}f(y,\tau_1)\mathrm{d}\Psi(y)\right)^{-1}\le x\right\}\end{aligned}$$

and

$$\begin{aligned}\lim_{t\to\infty}\mathrm{Ch}\left\{\frac{S_{\lfloor tx\rfloor+1}}{\lfloor tx\rfloor}>\frac{1}{x}\right\}&=1-\lim_{t\to\infty}\mathrm{Ch}\left\{\frac{\lfloor tx\rfloor+1}{\lfloor tx\rfloor}\cdot\frac{S_{\lfloor tx\rfloor+1}}{\lfloor tx\rfloor+1}\le\frac{1}{x}\right\}\\&=1-\mathcal{M}\left\{\int_{-\infty}^{+\infty}f(y,\tau_1)\mathrm{d}\Psi(y)\le\frac{1}{x}\right\}\\&=\mathcal{M}\left\{\left(\int_{-\infty}^{+\infty}f(y,\tau_1)\mathrm{d}\Psi(y)\right)^{-1}\le x\right\}.\end{aligned}$$

From the above three relations we get

$$\lim_{t\to\infty}\mathrm{Ch}\left\{\frac{S_{\lfloor tx\rfloor+1}}{\lfloor tx\rfloor+1}>\frac{t}{\lfloor tx\rfloor+1}\right\}=\mathcal{M}\left\{\left(\int_{-\infty}^{+\infty}f(y,\tau_1)\mathrm{d}\Psi(y)\right)^{-1}\le x\right\}$$

and then

$$\lim_{t\to\infty} \mathrm{Ch}\left\{\frac{N_t}{t} \le x\right\} = \mathcal{M}\left\{\left(\int_{-\infty}^{+\infty} f(y,\tau_1)\mathrm{d}\Psi(y)\right)^{-1} \le x\right\}.$$

The theorem is thus verified.

Exercise B.22: Let $\eta_1, \eta_2, \cdots$ be iid positive random variables, and let $\tau_1, \tau_2, \cdots$ be iid positive uncertain variables. Assume N_t is an uncertain random renewal process with interarrival times $\eta_1+\tau_1, \eta_2+\tau_2, \cdots$ Show that

$$\frac{N_t}{t} \to \frac{1}{E[\eta_1]+\tau_1} \tag{B.171}$$

in the sense of convergence in distribution as $t \to \infty$.

Exercise B.23: Let $\eta_1, \eta_2, \cdots$ be iid positive random variables, and let $\tau_1, \tau_2, \cdots$ be iid positive uncertain variables. Assume N_t is an uncertain random renewal process with interarrival times $\eta_1\tau_1, \eta_2\tau_2, \cdots$ Show that

$$\frac{N_t}{t} \to \frac{1}{E[\eta_1]\tau_1} \tag{B.172}$$

in the sense of convergence in distribution as $t \to \infty$.

Theorem B.34 *(Yao [255]) Let $\eta_1, \eta_2, \cdots$ be iid random interarrival times, and let $\tau_1, \tau_2, \cdots$ be iid uncertain rewards. Assume N_t is a stochastic renewal process with interarrival times $\eta_1, \eta_2, \cdots$ Then*

$$R_t = \sum_{i=1}^{N_t} \tau_i \tag{B.173}$$

is an uncertain random renewal reward process, and

$$\frac{R_t}{t} \to \frac{\tau_1}{E[\eta_1]} \tag{B.174}$$

in the sense of convergence in distribution as $t \to \infty$.

Proof: Let Υ denote the uncertainty distribution of τ_1. Then for each realization of N_t, the uncertain variable

$$\frac{1}{N_t}\sum_{i=1}^{N_t} \tau_i$$

follows the uncertainty distribution Υ. In addition, by the definition of chance distribution, we have

$$\begin{aligned}\mathrm{Ch}\left\{\frac{R_t}{t}\le x\right\} &= \int_0^1 \Pr\left\{\mathcal{M}\left\{\frac{R_t}{t}\le x\right\}\ge r\right\}\mathrm{d}r\\ &= \int_0^1 \Pr\left\{\mathcal{M}\left\{\frac{1}{N_t}\sum_{i=1}^{N_t}\tau_i\le\frac{tx}{N_t}\right\}\ge r\right\}\mathrm{d}r\\ &= \int_0^1 \Pr\left\{\Upsilon\left(\frac{tx}{N_t}\right)\ge r\right\}\mathrm{d}r\end{aligned}$$

for any real number x. Since N_t is a stochastic renewal process with iid interarrival times $\eta_1,\eta_2,\cdots$, we have

$$\frac{t}{N_t}\to E[\eta_1],\quad a.s.$$

as $t\to\infty$. It follows from the Lebesgue domain convergence theorem that

$$\begin{aligned}\lim_{t\to\infty}\mathrm{Ch}\left\{\frac{R_t}{t}\le x\right\} &= \lim_{t\to\infty}\int_0^1 \Pr\left\{\Upsilon\left(\frac{tx}{N_t}\right)\ge r\right\}\mathrm{d}r\\ &= \int_0^1 \Pr\left\{\Upsilon(E[\eta_1]x)\ge r\right\}\mathrm{d}r=\Upsilon(E[\eta_1]x)\end{aligned}$$

that is just the uncertainty distribution of $\tau_1/E[\eta_1]$. The theorem is thus proved.

Theorem B.35 *(Yao [255]) Let $\eta_1,\eta_2,\cdots$ be iid random rewards, and let $\tau_1,\tau_2,\cdots$ be iid uncertain interarrival times. Assume N_t is an uncertain renewal process with interarrival times $\tau_1,\tau_2,\cdots$ Then*

$$R_t=\sum_{i=1}^{N_t}\eta_i \tag{B.175}$$

is an uncertain random renewal reward process, and

$$\frac{R_t}{t}\to\frac{E[\eta_1]}{\tau_1} \tag{B.176}$$

in the sense of convergence in distribution as $t\to\infty$.

Proof: Let Υ denote the uncertainty distribution of τ_1. It follows from the definition of chance distribution that for any real number x, we have

$$\begin{aligned}\mathrm{Ch}\left\{\frac{R_t}{t}\le x\right\} &= \int_0^1 \Pr\left\{\mathcal{M}\left\{\frac{R_t}{t}\le x\right\}\ge r\right\}\mathrm{d}r\\ &= \int_0^1 \Pr\left\{\mathcal{M}\left\{\frac{1}{x}\cdot\frac{1}{N_t}\sum_{i=1}^{N_t}\eta_i\le\frac{t}{N_t}\right\}\ge r\right\}\mathrm{d}r.\end{aligned}$$

Since N_t is an uncertain renewal process with iid interarrival times $\tau_1, \tau_2, \cdots$, by using Theorem 13.3, we have

$$\frac{t}{N_t} \to \tau_1$$

in the sense of convergence in distribution as $t \to \infty$. In addition, for each realization of N_t, the law of large numbers for random variables says

$$\frac{1}{N_t}\sum_{i=1}^{N_t} \eta_i \to E[\eta_1], \quad a.s.$$

as $t \to \infty$ for each number x. It follows from the Lebesgue domain convergence theorem that

$$\lim_{t\to\infty} \mathrm{Ch}\left\{\frac{R_t}{t} \le x\right\} = \int_0^1 \Pr\left\{1 - \Upsilon\left(\frac{E[\eta_1]}{x}\right) \ge r\right\} \mathrm{d}r = 1 - \Upsilon\left(\frac{E[\eta_1]}{x}\right)$$

that is just the uncertainty distribution of $E[\eta_1]/\tau_1$. The theorem is thus proved.

B.14 Bibliographic Notes

Probability theory was developed by Kolmogorov [88] in 1933 for modeling frequencies, while uncertainty theory was founded by Liu [122] in 2007 for modeling belief degrees. However, in many cases, uncertainty and randomness simultaneously appear in a complex system. In order to describe this phenomenon, chance theory was pioneered by Liu [149] in 2013 with the concepts of uncertain random variable, chance measure and chance distribution. Liu [149] also proposed the concepts of expected value and variance of uncertain random variables. As an important contribution to chance theory, Liu [150] presented an operational law of uncertain random variables. In addition, Yao and Gao [251] verified a law of large numbers for uncertain random variables, and Hou [64] investigated the distance between uncertain random variables.

Stochastic programming was first studied by Dantzig [28] in 1965, while uncertain programming was first proposed by Liu [124] in 2009. In order to model optimization problems with not only uncertainty but also randomness, uncertain random programming was founded by Liu [150] in 2013. As extensions, Zhou, Yang and Wang [279] proposed uncertain random multiobjective programming for optimizing multiple, noncommensurable and conflicting objectives, Qin [191] proposed uncertain random goal programming in order to satisfy as many goals as possible in the order specified, and Ke [83] proposed uncertain random multilevel programming for studying decentralized decision systems in which the leader and followers may have their own decision variables and objective functions. After that, uncertain random programming was developed steadily and applied widely.

Probabilistic risk analysis was dated back to 1952 when Roy [199] proposed his safety-first criterion for portfolio selection. Another important contribution is the probabilistic value-at-risk methodology developed by Morgan [171] in 1996. On the other hand, uncertain risk analysis was proposed by Liu [128] in 2010 for evaluating the risk index that is the uncertain measure of an uncertain system being loss-positive. More generally, in order to quantify the risk of uncertain random systems, Liu and Ralescu [151] invented the tool of uncertain random risk analysis. Furthermore, value-at-risk methodology was presented by Liu and Ralescu [153] and expected loss was investigated by Liu and Ralescu [154] for dealing with uncertain random systems.

Probabilistic reliability analysis was traced back to 1944 when Pugsley [187] first proposed structural accident rates for the aeronautics industry. Nowadays, probabilistic reliability analysis has become a widely used discipline. As a new methodology, uncertain reliability analysis was developed by Liu [128] in 2010 for evaluating the reliability index. More generally, for dealing with uncertain random systems, Wen and Kang [232] presented the tool of uncertain random reliability analysis.

Random graph was defined by Erdős and Rényi [38] in 1959 and independently by Gilbert [56] at nearly the same time. As an alternative, uncertain graph was proposed by Gao and Gao [50] in 2013 via uncertainty theory. Assuming some edges exist with some degrees in probability measure and others exist with some degrees in uncertain measure, Liu [138] defined the concept of uncertain random graph in 2014.

Random network was first investigated by Frank and Hakimi [43] in 1965 for modeling communication network with random capacities. From then on, the random network was well developed and widely applied. As a breakthrough approach, uncertain network was first explored by Liu [129] in 2010 for modeling project scheduling problem with uncertain duration times. More generally, assuming some weights are random variables and others are uncertain variables, Liu [138] initialized the concept of uncertain random network in 2014.

One of the earliest investigations of stochastic process was Bachelier [3] in 1900, and the study of uncertain process was started by Liu [123] in 2008. In order to deal with uncertain random phenomenon evolving in time, Gao and Yao [47] presented an uncertain random process in the light of chance theory. Gao and Yao [47] also proposed an uncertain random renewal process. As extensions, Yao [255] discussed an uncertain random renewal reward process, and Yao [256] investigated an uncertain random alternating renewal process.

Appendix C

Frequently Asked Questions

This appendix will answer some frequently asked questions related to probability theory and uncertainty theory as well as their applications. This appendix will also show why fuzzy set is a wrong model in both theory and practice. Finally, I will clarify what uncertainty is.

C.1 What is the meaning that an object follows the laws of probability theory?

We say an object (e.g. frequency) follows the laws of probability theory if it meets not only the three axioms (Kolmogorov [88]) but also the product probability theorem of probability theory:

Axiom 1 *(Normality Axiom)* $\Pr\{\Omega\} = 1$ *for the universal set* Ω;

Axiom 2 *(Nonnegativity Axiom)* $\Pr\{A\} \geq 0$ *for any event* A;

Axiom 3 *(Additivity Axiom) For every countable sequence of mutually disjoint events* $A_1, A_2, \cdots$, *we have*

$$\Pr\left\{\bigcup_{i=1}^{\infty} A_i\right\} = \sum_{i=1}^{\infty} \Pr\{A_i\}; \tag{C.1}$$

Theorem *(Product Probability) Let* $(\Omega_k, \mathcal{A}_k, \Pr_k)$ *be probability spaces for* $k = 1, 2, \cdots$ *Then there is a unique probability measure* $\Pr$ *such that*

$$\Pr\left\{\prod_{k=1}^{\infty} A_k\right\} = \prod_{k=1}^{\infty} \Pr_k\{A_k\} \tag{C.2}$$

where A_k *are arbitrarily chosen events from* $\mathcal{A}_k$ *for* $k = 1, 2, \cdots$, *respectively.*

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5

It is easy for us to understand why we need to justify that the object meets the three axioms. However, some readers may wonder why we need to justify that the object meets the product probability theorem. The reason is that *product probability theorem cannot be deduced from Kolmogorov's axioms* except we presuppose that the product probability meets the three axioms. Would that surprise you? In fact, the same probability theory may be derived if the product probability theorem was replaced with an axiom:

Axiom 4 *(Product Probability) Let* $(\Omega_k, \mathcal{A}_k, \mathrm{Pr}_k)$ *be probability spaces for* $k = 1, 2, \cdots$ *The product probability measure* Pr *is a probability measure satisfying*

$$\Pr\left\{\prod_{k=1}^{\infty} A_k\right\} = \prod_{k=1}^{\infty} \Pr_k\{A_k\} \tag{C.3}$$

where A_k *are arbitrarily chosen events from* $\mathcal{A}_k$ *for* $k = 1, 2, \cdots$*, respectively.*

One advantage of this revision is to force the practitioners to justify the product probability for their own problems. Please keep in mind that "an object follows the laws of probability theory" is equivalent to "it meets the four axioms of probability theory", or "it meets the three axioms plus the product probability theorem". This assertion is stronger than "an object meets the three axioms of Kolmogorov". In other words, the three axioms do not ensure that an object follows the laws of probability theory.

There exist two broad categories of interpretations of probability, one is *frequency interpretation* and the other is *belief interpretation*. The frequency interpretation takes the probability to be the frequency with which an event happens (Venn [224], Reichenbach [197], von Mises [225]), while the belief interpretation takes the probability to be the degree to which we believe an event will happen (Ramsey [196], de Finetti [31], Savage [202]).

The debate between different interpretations has been lasting from the nineteenth century. Personally, I agree with the frequency interpretation, but strongly oppose the belief interpretation of probability because frequency follows the laws of probability theory but belief degree does not. The detailed reasons will be given in the following a few sections.

C.2 Why does frequency follow the laws of probability theory?

In order to show that the frequency follows the laws of probability theory, we must verify that the frequency meets not only the three axioms of Kolmogorov but also the product probability theorem.

First, the frequency of the universal set takes value 1 because the universal set always happens. Thus the frequency meets the normality axiom. Second, it is obvious that the frequency is a number between 0 and 1. Thus the frequency of any event is nonnegative, and the frequency meets the nonnegativity axiom. Third, for any disjoint events A and B, if A happens α times and B happens β times, it is clear that the union $A \cup B$ happens $\alpha + \beta$ times. This means the frequency is additive and then meets the additivity axiom. Finally, numerous experiments showed that if A and B are two events from different probability spaces (essentially they come from two different experiments) and happen α and β times (in percentage), respectively, then the product $A \times B$ happens $\alpha \times \beta$ times. See Figure C.1. Thus the frequency meets the product probability theorem. Hence the frequency does follow the laws of probability theory. In fact, frequency is the only empirical basis for probability theory.

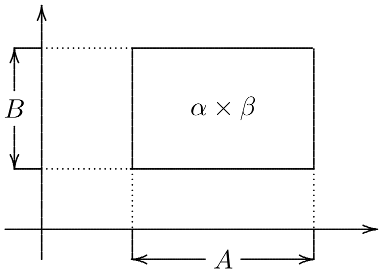


Figure C.1: Let A and B be two events from different probability spaces (essentially they come from two different experiments). If A happens α times and B happens β times, then the product $A \times B$ happens $\alpha \times \beta$ times, where α and β are understood as percentage numbers.

C.3 Why is probability theory unable to model belief degree?

In order to obtain the belief degree of some event, the decision maker needs to launch a consultation process with a domain expert. The decision maker is the user of belief degree while the domain expert is the holder. For justifying whether probability theory is able to model belief degree or not, we must check if the belief degree follows the laws of probability theory.

First, "1" means "complete belief" and we cannot be in more belief than "complete belief". This means the belief degree of any event cannot exceed 1. In particular, the belief degree of the universal set takes value 1 because it is completely believable. Hence the belief degree meets the normality axiom of probability theory.

Second, the belief degree meets the nonnegativity axiom because "0" means "complete disbelief" and we cannot be in more disbelief than "complete disbelief".

Third, de Finetti [31] interpreted the belief degree of an event as the fair betting ratio (price/stake) of a bet that offers $1 if the event happens and nothing otherwise. For example, if the domain expert thinks the belief degree of an event A is α, then the price of the bet about A is $\alpha \times 100$¢. Here the word "fair" means both the domain expert and the decision maker are willing to either buy or sell this bet at this price.

Besides, Ramsey [196] suggested a Dutch book argument[1] that says *the belief degree is irrational if there exists a book that guarantees either the domain expert or the decision maker a loss.* For the moment, we are assumed to agree with it.

Let A_1 be a bet that offers \$1 if A_1 happens, and let A_2 be a bet that offers \$1 if A_2 happens. Assume the belief degrees of A_1 and A_2 are α_1 and α_2, respectively. This means the prices of A_1 and A_2 are $\$\alpha_1$ and $\$\alpha_2$, respectively. Now we consider the bet $A_1 \cup A_2$ that offers \$1 if either A_1 or A_2 happens, and write the belief degree of $A_1 \cup A_2$ by α. This means the price of $A_1 \cup A_2$ is $\$\alpha$. If $\alpha > \alpha_1 + \alpha_2$, then you (i) sell A_1, (ii) sell A_2, and (iii) buy $A_1 \cup A_2$. It is clear that you are guaranteed to lose $\alpha - \alpha_1 - \alpha_2 > 0$. Thus there exists a Dutch book and the assumption $\alpha > \alpha_1 + \alpha_2$ is irrational. If $\alpha < \alpha_1 + \alpha_2$, then you (i) buy A_1, (ii) buy A_2, and (iii) sell $A_1 \cup A_2$. It is clear that you are guaranteed to lose $\alpha_1 + \alpha_2 - \alpha > 0$. Thus there exists a Dutch book and the assumption $\alpha < \alpha_1 + \alpha_2$ is irrational. Hence we have to assume $\alpha = \alpha_1 + \alpha_2$ and the belief degree meets the additivity axiom (but this assertion is questionable because you cannot reverse "buy" and "sell" arbitrarily due to the unequal status of the decision maker and the domain expert).

Until now we have verified that the belief degree meets the three axioms of probability theory. Almost all subjectivists stop here and assert that belief degree follows the laws of probability theory. Unfortunately, the evidence is not enough for this conclusion because we have not verified whether the belief degree meets the product probability theorem or not.

Recall the example of truck-cross-over-bridge on Page 6. Let A_i represent that the ith bridge strengths are greater than 90 tons, $i = 1, 2, \cdots, 50$, respectively. For each i, since your belief degree for A_i is 75%, you are willing to pay 75¢ for the bet that offers \$1 if A_i happens. If the belief degree did follow the laws of probability theory, then it would be fair to pay

$$\underbrace{75\% \times 75\% \times \cdots \times 75\%}_{50} \times 100¢ \approx 0.00006¢ \tag{C.4}$$

for a bet that offers \$1 if $A_1 \times A_2 \times \cdots \times A_{50}$ happens. Notice that the odd is over a million and $A_1 \times A_2 \times \cdots \times A_{50}$ definitely happens because the real strengths of the 50 bridges are assumed to range from 95 to 110 tons. All

[1] A Dutch book in a betting market is a set of bets which guarantees a loss, regardless of the outcome of the gamble. For example, let A be a bet that offers \$1 if A happens, let B be a bet that offers \$1 if B happens, and let $A \vee B$ be a bet that offers \$1 if either A or B happens. If the prices of A, B and $A \vee B$ are 30¢, 40¢ and 80¢, respectively, and you (i) sell A, (ii) sell B, and (iii) buy $A \vee B$, then you are guaranteed to lose 10¢ no matter what happens. Thus there exists a Dutch book, and the prices are considered to be irrational.

of us will be happy to bet on it. But who is willing to offer such a bet? It seems that no one does, and then the belief degree of $A_1 \times A_2 \times \cdots \times A_{50}$ is not the product of each individuals. Hence the belief degree does not follow the laws of probability theory.

It is thus concluded that the belief interpretation of probability is unacceptable. The main mistake of subjectivists is that they only justify the belief degree meets the three axioms of probability theory, but do not check if it meets the product probability theorem.

C.4 Why should belief degree be understood as an oddsmaker's betting ratio rather than a fair one?

There are many similarities between a betting market and a consultation process. First, the oddsmaker and the bettor are two sides in the betting market, while the domain expert and the decision maker are two sides in the consultation process. Second, the oddsmaker is the maker of betting ratio while the domain expert is the holder of belief degree. Third, the bettor is the buyer of bets while the decision maker is the user of belief degrees. Fourth, the oddsmaker wants to get a commission while the domain expert is conservatism.

The status of the domain expert and the decision maker is unequal and they cannot exchange the roles with each other. Because of the conservatism, the human beings usually overweight unlikely events. Thus the decision maker cannot expect the domain expert provides a "fair" belief degree just like that the bettor cannot expect the oddsmaker provides a "fair" betting ratio ("fair" implies the sum of the betting ratios of all outcomes is just 1, but the real sum is usually between 1.1 and 1.3). This is the reason why I do not agree with de Finetti on fair betting ratio.

Instead, I think *the belief degree should be understood as an oddsmaker's betting ratio* that is not "fair" at all. The bettor (decision maker) can choose to buy the bets from the oddsmaker (domain expert) but cannot sell them at this prices. Meanwhile, the oddsmaker (domain expert) sells the bets to the bettor (domain expert) but never buys them. In other words, the bettor (decision maker) is always a buyer while the oddsmaker (domain expert) is always a seller.

The oddsmaker is never willing to accept a negative commission. Therefore, I would like to suggest a negative commission argument that says *the belief degree is irrational if there exists a book that guarantees the oddsmaker (domain expert) a loss.* Keep in mind that the decision maker and the domain expert cannot exchange their roles due to the unequal status of them. It is thus concluded that the belief degree is considered to be irrational if it makes the domain expert accept a sure-loss book.

C.5 Why does belief degree follow the laws of uncertainty theory?

In order to justify the belief degree follows the laws of uncertainty theory, we must show that it meets the four axioms of uncertainty theory (Liu [122][125]):

Axiom 1 *(Normality Axiom)* $\mathcal{M}\{\Gamma\} = 1$ *for the universal set* Γ;

Axiom 2 *(Duality Axiom)* $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any event* Λ;

Axiom 3 *(Subadditivity Axiom) For every countable sequence of events* $\Lambda_1, \Lambda_2, \cdots$, *we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}; \tag{C.5}$$

Axiom 4 *(Product Axiom) Let* $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ *be uncertainty spaces for* $k = 1, 2, \cdots$ *The product uncertain measure* $\mathcal{M}$ *is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\} \tag{C.6}$$

where Λ_k *are arbitrarily chosen events from* $\mathcal{L}_k$ *for* $k = 1, 2, \cdots$, *respectively.*

First, "1" means "complete belief" and we cannot be in more belief than "complete belief". This means the belief degree of any event cannot exceed 1. In particular, the belief degree of the universal set takes value 1 because it is completely believable. Thus the belief degree meets the normality axiom of uncertainty theory.

Second, the law of truth conservation says the belief degrees of an event and its negation sum to unity. For example, if a domain expert says an event is true with belief degree α, then all of us will think that the event is false with belief degree $1 - \alpha$. The belief degree is considered to be irrational if it violates the law of truth conservation. Thus the belief degree meets the duality axiom. In practice, this law is easy for human beings to obey.

Third, let Λ_1 be a bet that offers \$1 if Λ_1 happens, and let Λ_2 be a bet that offers \$1 if Λ_2 happens. Assume the belief degrees of Λ_1 and Λ_2 are α_1 and α_2, respectively. This means the prices of Λ_1 and Λ_2 are $\$\alpha_1$ and $\$\alpha_2$, respectively. Now we consider the bet $\Lambda_1 \cup \Lambda_2$ that offers \$1 if either Λ_1 or Λ_2 happens, and write the belief degree of $\Lambda_1 \cup \Lambda_2$ by α. It follows from the duality axiom that the belief degree of $(\Lambda_1 \cup \Lambda_2)^c$ is $1 - \alpha$. This means the price of the bet about $(\Lambda_1 \cup \Lambda_2)^c$ (i.e., a bet that offers \$1 if neither Λ_1 nor Λ_2 happens) is $\$(1 - \alpha)$. If $\alpha > \alpha_1 + \alpha_2$, then the decision maker buys Λ_1, Λ_2 and $(\Lambda_1 \cup \Lambda_2)^c$ from the domain expert. It is clear the domain expert is guaranteed to lose

$$1 - \alpha_1 - \alpha_2 - (1 - \alpha) = \alpha - \alpha_1 - \alpha_2 > 0. \tag{C.7}$$

It follows from the negative commission argument that the assumption $\alpha > \alpha_1 + \alpha_2$ is irrational. Hence we have to assume $\alpha \le \alpha_1 + \alpha_2$ and the belief degree meets the subadditivity axiom. Note that the decision maker cannot sell the bets to the domain expert due to the unequal status of them.

Finally, regarding the product axiom, let us recall the example of truck-cross-over-bridge on Page 6. Suppose A_i represent that the ith bridge strengths are greater than 90 tons, $i = 1, 2, \cdots, 50$, respectively. For each i, since the belief degree of A_i is 75%, the price of the bet about A_i is 75¢. It is reasonable to pay

$$\underbrace{75¢ \wedge 75¢ \wedge \cdots \wedge 75¢}_{50} = 75¢ \tag{C.8}$$

for a bet that offers \$1 if $A_1 \times A_2 \times \cdots \times A_{50}$ happens. Thus the belief degree meets the product axiom of uncertainty theory.

Hence the belief degree follows the laws of uncertainty theory. It is easy to prove that if a set of belief degrees violate the laws of uncertainty theory, then there exists a book that guarantees the domain expert a loss. It is also easy to prove that if a set of belief degrees follow the laws of uncertainty theory, then there does not exist any book that guarantees the domain expert a loss.

C.6 What is the difference between probability theory and uncertainty theory?

The main difference between probability theory (Kolmogorov [88]) and uncertainty theory (Liu [122]) is that the probability measure of a product of events is the product of the probability measures of the individual events, i.e.,

$$\Pr\{A \times B\} = \Pr\{A\} \times \Pr\{B\}, \tag{C.9}$$

and the uncertain measure of a product of events is the minimum of the uncertain measures of the individual events, i.e.,

$$\mathcal{M}\{A \times B\} = \mathcal{M}\{A\} \wedge \mathcal{M}\{B\}. \tag{C.10}$$

This difference implies that random variables and uncertain variables obey different operational laws.

Probability theory and uncertainty theory are complementary mathematical systems that provide two acceptable mathematical models to deal with the indeterminate world. Probability is interpreted as frequency, while uncertainty is interpreted as personal belief degree.

C.7 What goes wrong with Cox's theorem?

Some people affirm that *probability theory is the only legitimate approach.* Perhaps this misconception is rooted in Cox's theorem [26] that any measure of belief is "isomorphic" to a probability measure. However, uncertain measure is considered coherent but not isomorphic to any probability measure. What goes wrong with Cox's theorem? Personally I think that Cox's theorem presumes the truth value of conjunction $P \wedge Q$ is a twice differentiable function f of truth values of the two propositions P and Q, i.e.,

$$T(P \wedge Q) = f(T(P), T(Q)) \tag{C.11}$$

and then excludes uncertain measure from its start because the function $f(x, y) = x \wedge y$ used in uncertainty theory is not differentiable with respect to x and y. In fact, there does not exist any evidence that the truth value of conjunction is completely determined by the truth values of individual propositions, let alone a twice differentiable function.

On the one hand, it is recognized that probability theory is a legitimate approach to deal with the frequency. On the other hand, at any rate, it is impossible that probability theory is the unique one for modeling indeterminacy. In fact, it has been demonstrated in this book that uncertainty theory is successful to deal with belief degrees.

C.8 What is the difference between possibility theory and uncertainty theory?

The essential difference between possibility theory (Zadeh [263]) and uncertainty theory (Liu [122]) is that the former assumes

$$\text{Pos}\{A \cup B\} = \text{Pos}\{A\} \vee \text{Pos}\{B\} \tag{C.12}$$

for any events A and B no matter if they are independent or not, and the latter holds

$$\mathcal{M}\{A \cup B\} = \mathcal{M}\{A\} \vee \mathcal{M}\{B\} \tag{C.13}$$

only for independent events A and B. A lot of surveys showed that the measure of a union of events is usually greater than the maximum of the measures of individual events when they are not independent. This fact states that human brains do not behave fuzziness.

Both uncertainty theory and possibility theory attempt to model belief degrees, where the former uses the tool of uncertain measure and the latter uses the tool of possibility measure. Thus they are complete competitors.

C.9 Why is fuzzy variable unable to model indeterminate quantity?

A fuzzy variable is a function from a possibility space to the set of real numbers (Nahmias [172]). Some people think that fuzzy variable is a suitable tool for modeling indeterminate quantity. Is it really true? Unfortunately, the answer is negative.

Let us reconsider the counterexample of truck-cross-over-bridge (Liu [131]). If the bridge strength is regarded as a fuzzy variable ξ, then we may assign it a membership function, say

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 80 \\ (x-80)/10, & \text{if } 80 \le x \le 90 \\ 1, & \text{if } 90 \le x \le 110 \\ (120-x)/10, & \text{if } 110 \le x \le 120 \\ 0, & \text{if } x \ge 120 \end{cases} \tag{C.14}$$

that is just the trapezoidal fuzzy variable $(80, 90, 110, 120)$. Please do not argue why I choose such a membership function because it is not important for the focus of debate. Based on the membership function μ and the definition of possibility measure

$$\text{Pos}\{\xi \in B\} = \sup_{x \in B} \mu(x), \tag{C.15}$$

it is easy for us to infer that

$$\text{Pos}\{\text{“bridge strength”} = 100\} = 1, \tag{C.16}$$

$$\text{Pos}\{\text{“bridge strength”} \ne 100\} = 1. \tag{C.17}$$

Thus we immediately conclude the following three propositions:

(a) the bridge strength is “exactly 100 tons” with possibility measure 1,

(b) the bridge strength is “not 100 tons” with possibility measure 1,

(c) “exactly 100 tons” is as possible as “not 100 tons”.

The first proposition says we are 100% sure that the bridge strength is “exactly 100 tons”, neither less nor more. What a coincidence it should be! It is doubtless that the belief degree of “exactly 100 tons” is almost zero, and nobody is so naive to expect that “exactly 100 tons” is the true bridge strength. The second proposition sounds good. The third proposition says “exactly 100 tons” and “not 100 tons” have the same possibility measure. Thus we have to regard them “equally likely”. Consider a bet: you get $1 if the bridge strength is “exactly 100 tons”, and pay $1 if the bridge strength is“not 100 tons”. Do you think the bet is fair? It seems that no one thinks so. Hence the conclusion (c) is unacceptable because “exactly 100 tons” is almost impossible compared with “not 100 tons”. This paradox shows that those indeterminate quantities like the bridge strength cannot be quantified by possibility measure and then they are not fuzzy concepts.

C.10 Why is fuzzy set unable to model unsharp concept?

A fuzzy set is defined by its membership function μ which assigns to each element x a real number $\mu(x)$ in the interval $[0,1]$, where the value of $\mu(x)$

represents the grade of membership of x in the fuzzy set. This definition was given by Zadeh [260] in 1965. Although I strongly respect Professor Lotfi Zadeh's achievements, I disagree with him on the topic of fuzzy set.

Up to now, fuzzy set theory has not evolved as a mathematical system because of its inconsistence. Theoretically, it is undeniable that there exist too many contradictions in fuzzy set theory. In practice, perhaps some people believe that fuzzy set is a suitable tool to model unsharp concepts. Unfortunately, it is not true. In order to convince the reader, let us examine the concept of "young". Without loss of generality, assume "young" has a trapezoidal membership function $(15, 20, 30, 40)$, i.e.,

$$\mu(x) = \begin{cases} 0, & \text{if } x \le 15 \\ (x-15)/5, & \text{if } 15 \le x \le 20 \\ 1, & \text{if } 20 \le x \le 30 \\ (40-x)/10, & \text{if } 30 \le x \le 40 \\ 0, & \text{if } x \ge 40. \end{cases} \tag{C.18}$$

It follows from the fuzzy set theory that "young" takes any values of α-cut of μ, and then we infer that

$$\text{Pos}\{[20\text{yr}, 30\text{yr}] \subset \text{“young”}\} = 1, \tag{C.19}$$

$$\text{Pos}\{\text{“young”} \subset [20\text{yr}, 30\text{yr}]\} = 1. \tag{C.20}$$

Thus we immediately conclude two propositions:

(a) "young" includes [20yr, 30yr] with possibility measure 1,

(b) "young" is included in [20yr, 30yr] with possibility measure 1.

The first proposition sounds good. However, the second proposition seems unacceptable because the belief degree that "young" is between 20yr to 30yr is impossible to achieve up to 1 (in fact, the belief degree should be almost 0 due to the fact that 19yr and 31yr are also nearly sure to be "young"). This result says that "young" cannot be regarded as a fuzzy set.

C.11 Does the stock price follow stochastic differential equation or uncertain differential equation?

The origin of stochastic finance theory can be traced to Louis Bachelier's doctoral dissertation *Théorie de la Speculation* in 1900. However, Bachelier's work had little impact for more than a half century. After Kiyosi Ito invented stochastic calculus [66] in 1944 and stochastic differential equation [67] in 1951, stochastic finance theory was well developed among others by Samuelson [201], Black and Scholes [8] and Merton [168] during the 1960s and 1970s.

Traditionally, stochastic finance theory presumes that the stock price (including interest rate and currency exchange rate) follows Ito's stochastic differential equation. Is it really reasonable? In fact, this widely accepted presumption was continuously challenged by many scholars.

As a paradox given by Liu [134], let us assume that the stock price X_t follows the stochastic differential equation,

$$\mathrm{d}X_t = eX_t\mathrm{d}t + \sigma X_t\mathrm{d}W_t \tag{C.21}$$

where e is the log-drift, σ is the log-diffusion, and W_t is a Wiener process. Let us see what will happen with such an assumption. It follows from the stochastic differential equation (C.21) that X_t is a geometric Wiener process, i.e.,

$$X_t = X_0 \exp((e - \sigma^2/2)t + \sigma W_t) \tag{C.22}$$

from which we derive

$$W_t = \frac{\ln X_t - \ln X_0 - (e - \sigma^2/2)t}{\sigma} \tag{C.23}$$

whose increment is

$$\Delta W_t = \frac{\ln X_{t+\Delta t} - \ln X_t - (e - \sigma^2/2)\Delta t}{\sigma}. \tag{C.24}$$

Write

$$A = -\frac{(e - \sigma^2/2)\Delta t}{\sigma}. \tag{C.25}$$

Note that the stock price X_t is actually a step function of time with a finite number of jumps although it looks like a curve. During a fixed period (e.g. one week), without loss of generality, we assume that X_t is observed to have 100 jumps. Now we divide the period into 10000 equal intervals. Then we may observe 10000 samples of X_t. It follows from (C.24) that ΔW_t has 10000 samples that consists of 9900 A's and 100 other numbers:

$$\underbrace{A,\ A,\cdots,A,}_{9900}\quad \underbrace{B,\ C,\cdots,Z.}_{100} \tag{C.26}$$

Nobody can believe that those 10000 samples follow a normal probability distribution with expected value 0 and variance Δt. This fact is in contradiction with the property of Wiener process that the increment ΔW_t is a normal random variable. Therefore, the real stock price X_t does not follow the stochastic differential equation.

Perhaps some people think that the stock price does behave like a geometric Wiener process (or Ornstein-Uhlenbeck process) in macroscopy although they recognize the paradox in microscopy. However, as the very core of stochastic finance theory, Ito's calculus is just built on the microscopic structure (i.e., the differential $\mathrm{d}W_t$) of Wiener process rather than macroscopic

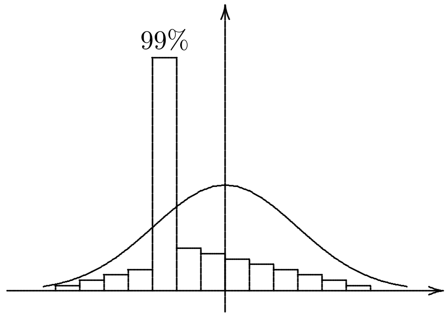


Figure C.2: There does not exist any continuous probability distribution (curve) that can approximate to the frequency (histogram) of ΔW_t. Hence it is impossible that the real stock price X_t follows any Ito's stochastic differential equation.

structure. More precisely, Ito's calculus is dependent on the presumption that $\mathrm{d}W_t$ is a normal random variable with expected value 0 and variance $\mathrm{d}t$. This unreasonable presumption is what causes the second order term in Ito's formula,

$$\mathrm{d}X_t = \frac{\partial h}{\partial t}(t, W_t)\mathrm{d}t + \frac{\partial h}{\partial w}(t, W_t)\mathrm{d}W_t + \frac{1}{2}\frac{\partial^2 h}{\partial w^2}(t, W_t)\mathrm{d}t. \tag{C.27}$$

In fact, the increment of stock price is impossible to follow any continuous probability distribution.

On the basis of the above paradox, personally I do not think Ito's calculus can play the essential tool of finance theory because Ito's stochastic differential equation is impossible to model stock price. As a substitute, uncertain calculus may be a potential mathematical foundation of finance theory. We will have a theory of uncertain finance if the stock price, interest rate and exchange rate are assumed to follow uncertain differential equations.

C.12 How did "uncertainty" evolve over the past 100 years?

After the word "randomness" was used to represent any probabilistic phenomena, Knight (1921) and Keynes (1936) started to use the word "uncertainty" to represent any non-probabilistic phenomena. The academic community also calls it Knightian uncertainty, Keynesian uncertainty, or true uncertainty. Unfortunately, it seems impossible for us to develop a mathematical theory to deal with such a broad class of uncertainty because "non-probability" represents too many things. This disadvantage makes uncertainty not able to become a scientific terminology. Despite that, it is recognized that Knight and Keynes made a great process to break the monopoly of probability theory.

However, a major retrogression arose from Cox (1946) with a theorem that human's belief degree is isomorphic to a probability measure. Many people do not notice that Cox's theorem is based on an unreasonable assumption, and then mistakenly believe that uncertainty and probability are synonymous. This idea remains alive today under the name of subjective probability (de Finetti, 1937). Yet numerous experiments demonstrated that the belief degree does not follow the laws of probability theory.

An influential exploration by Zadeh (1965) was the fuzzy set theory that was widely said to be successfully applied in many areas of our life. However, fuzzy set theory has neither evolved as a mathematical system nor become a suitable tool for rationally modeling belief degrees. The main mistake of fuzzy set theory is based on the wrong assumption that the belief degree of a union of events is the maximum of the belief degrees of the individual events no matter if they are independent or not. A lot of surveys showed that human brains do not behave fuzziness in the sense of Zadeh.

The latest development was uncertainty theory founded by Liu (2007). Nowadays, uncertainty theory has become a branch of pure mathematics that is not only a formal study of an abstract structure (i.e., uncertainty space) but also applicable to modeling belief degrees. Perhaps some readers may complain that I never clarify what uncertainty is. I think we can answer it this way. Mathematically, *uncertainty is anything that follows the laws of uncertainty theory.* Practically, *uncertainty is anything that is described by belief degrees.* From then on, "uncertainty" became a scientific terminology on the basis of uncertainty theory.

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List of Frequently Used Symbols

$\mathcal{M}$	uncertain measure
$(\Gamma, \mathcal{L}, \mathcal{M})$	uncertainty space
ξ, η, τ	uncertain variables
Φ, Ψ, Υ	uncertainty distributions
Φ^{-1}, Ψ^{-1}, Υ^{-1}	inverse uncertainty distributions
μ, ν, λ	membership functions
μ^{-1}, ν^{-1}, λ^{-1}	inverse membership functions
$\mathcal{L}(a, b)$	linear uncertain variable
$\mathcal{Z}(a, b, c)$	zigzag uncertain variable
$\mathcal{N}(e, \sigma)$	normal uncertain variable
$\mathcal{LOGN}(e, \sigma)$	lognormal uncertain variable
(a, b, c)	triangular uncertain set
(a, b, c, d)	trapezoidal uncertain set
E	expected value
V	variance
H	entropy
X_t, Y_t, Z_t	uncertain processes
C_t	Liu process
N_t	renewal process
$\mathcal{Q}$	uncertain quantifier
$(\mathcal{Q}, S, P)$	uncertain proposition
$\vee$	maximum operator
$\wedge$	minimum operator
$\neg$	negation symbol
$\forall$	universal quantifier
$\exists$	existential quantifier
Pr	probability measure
$(\Omega, \mathcal{A}, \Pr)$	probability space
Ch	chance measure
k-max	the kth largest value
k-min	the kth smallest value
$\emptyset$	the empty set
$\Re$	the set of real numbers
iid	independent and identically distributed

B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5

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B. Liu, *Uncertainty Theory*, Springer Uncertainty Research,
DOI 10.1007/978-3-662-44354-5